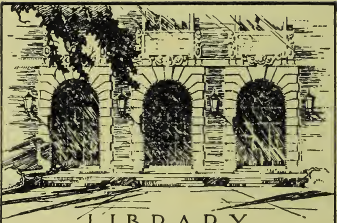

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Finsbury Technical Manuals

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PROFESSOR SILVANUS P. THOMPSON

D.Sc., B.A., F.R.S., M.I.E.E., &c.

AN ELEMENTARY TREATISE
ON
THE CALCULUS

FOR ENGINEERING STUDENTS

WITH NUMEROUS EXAMPLES AND PROBLEMS
WORKED OUT

BY

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PREFACE TO SECOND EDITION.

The present edition of this book has been carefully revised, and it is hoped that most of the errors, which are apt to occur in the first edition of a mathematical book, have been eliminated. An important alteration has been made, which commences on page 44, where the relation between electromotive-force, resistance, inductance, &c., in an electric circuit has been shown graphically.

The formulæ for the displacement, velocity, and acceleration of the piston of a steam engine, for any crank angle, have alsobeen added.

Many additional examples have been given at the end of some of the chapters.

In the Appendix will be found the method of finding the centre of areas and centre of mass both in rectangular and polar co-ordinates, the theorem of parallel axes, pressure on an immersed area, and depth of centre of pressure, vibrations natural and forced, which it is hoped will make the book more complete.

The author takes this opportunity to thank many readers for their suggestions, which have been acted upon as far as possible.

J. G.

Technical College,
Leonard Street, E.C., 1900.

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PREFACE.

It is generally admitted that there are very few books, if any, published on the Calculus which are suitable for the requirements of engineering students.

There are many excellent works on the subject, but, unfortunately, most of them are too advanced, and contain more matter of a purely mathematical character than the average engineering student requires. It has, therefore, been my aim in writing this book to put before the student as much, as space will permit, of the subject as he may require in actual practice. I have endeavoured to present the matter in as simple and practical a manner as possible, illustrating each part by examples fully worked out.

To Professor J. Perry, M.E., D.Sc., F.R.S., I am entirely indebted for the greater number of the practical problems and examples, and also for the practical way in which Fourier's Expansion is treated. His lines of teaching have been adopted as far as the subject is treated, although justice may not have been done to his methods.

In compiling this treatise I have consulted the works of Boole, Todhunter and Williamson.

My thanks are due to my colleagues and friends for valuable assistance rendered by way of checking examples, &c.

Hints or suggestions by the reader will be considered a favour.

J. GRAHAM.

Technical College, Finsbury:
March 1896.

CONTENTS.

CHAPTER I.

PAGE
Functions, rational and transcendental 1
Relation between function and variable 2
Representation of functions and differential coefficients 3
First derived function 4
Expressions for velocity and acceleration 5
Differentiation and neglected terms 5
Differential coefficient of a power 8
" " the algebraic sum of any number of functions 9
" " the product of two or more functions .. 11
" " a quotient 14
Differentiation of a function of a function 16
" the trigonometrical functions 18
" the inverse trigonometrical functions 23
" logarithmic and exponential functions 26
List of differential coefficients 33

CHAPTER II.

Successive derived functions and differentials 40
The nth derived functions of sin $k x$, cos $k x$, $\log_a x$ and e^{kx} cos $r x$ .. 42
Curves showing phase differences 45
Displacement, velocity, and acceleration of engine piston 46

CHAPTER III.

Taylor's and Maclaurin's theorems 50
Exponential theorem 56
Euler's formulæ for sine and cosine 58
Hyperbolic functions 58
Formulæ for calculating sine, cosine and tangent of any angle .. 61

CHAPTER IV.

PAGE
Indeterminate forms 64

CHAPTER V.

Maxima and minima of functions of one variable 70
Mechanical and electrical problems 76

CHAPTER VI.

Differential coefficient of functions of two or more variables 85
Implicit functions 88

CHAPTER VII.

Maxima and minima of functions of two or more variables 92
Undetermined multipliers 96

CHAPTER VIII.

Tangents and normals 105
Subtangents and subnormals 108

CHAPTER IX.

Radius of curvature of plane curves 112
Curvature 114
Curvature of the catenary 116
Centre of curvature 118

CHAPTER X.

Cycloid 121
Epicycloid 122
Hypocycloid 123

CHAPTER XI.

Integration 125
List of integrals 125
Integration by substitution 127
Typical examples on integration 127

CHAPTER XII.

PAGE
Integration by parts 137

CHAPTER XIII.

Integration of rational fractions 141
Partial fractions 141

CHAPTER XIV.

Integration by successive reduction 150

CHAPTER XV.

Definite integrals 157

CHAPTER XVI.

Areas of plane curves 160
The catenary 164
Areas in terms of polar co-ordinates 166

CHAPTER XVII.

Areas of surfaces of revolution 169
Guldinus' theorems 173

CHAPTER XVIII.

Volumes of solids of revolution 176

CHAPTER XIX.

Lengths of curves 180

CHAPTER XX

Fourier's expansion 184
Development of periodic functions.. 187

CHAPTER XXI.

PAGE
Differential equations 194
Solution of $\frac{dy}{dx} + A y = B$ 199
Solution of $V = R C + L \frac{dC}{dt}$, given $V = \sin n t$, R and L, to find C 201

CHAPTER XXII.

Solution of linear differential equations with constant coefficients .. 204

CHAPTER XXIII.

Symbolical method of solving linear differential equations 210

CHAPTER XXIV.

Law for the slipping of a belt on a pulley 216
Law for the thickness of a long pump-rod of uniform strength .. 217
Shape and deflection of girders loaded and supported in six different ways 218
Rule for the strength of a thick cylindric pipe 226
Moment of inertia of a rectangle 228
" " circle 229
" " cylinder 230
" " sphere 230

CHAPTER XXV.

Law for the discharge of a condenser with no self-induction 232
Law for the growth of a current in a circuit with self-induction .. 233
To destroy effect of self-induction by means of a condenser shunt .. 234
Discharge of a condenser with self-induction in the circuit 236
To diminish the idle current in the primary coil of a transformer .. 240
The Ferranti effect 242

MISCELLANEOUS EXAMPLES 244
APPENDIX 257

THE

DIFFERENTIAL CALCULUS.

CHAPTER I.

Art. I.—**Function.**

When two quantities are so related that if one undergoes any change in value the other undergoes a corresponding change, one of the quantities is said to be a function of the other. For instance, the area of a triangle, of constant height, varies directly as its base; therefore its area is a function of its base. The intensity of light varies inversely as the square of the distance, and we say that the intensity is a function of the distance. The surface of a sphere varies directly as the square of its diameter; therefore its surface is a function of its diameter.

Any analytical expression involving a variable quantity is said to be a function of that variable.

The algebraic expressions—x, $x + a$, $3x - 4$, x^2, $ax^3 + b$, $(x - c)^2$, $\sqrt{a^2 - x^2}$, $\frac{a + x}{b - x}$, are all *algebraic functions* of x.

The expressions, $\sin x$, $\cos x$, $\tan x$, $\log x$, e^x, $a e^{ax}$, are said to be *transcendental functions* of x; since they do not admit of being expressed in finite terms.

Functions are usually denoted by prefixing one of the characters f, ϕ, ψ, &c., to the variable. Thus $f(x)$, $\phi(x)$, $\psi(x)$, &c., may represent functions of the variable x. For instance, instead of $3x^2 - 4x + 9$, we may denote this expression by $f(x)$.

ART. 2.—Relation between Function and Variable.

Let us take, for example, the function $\frac{x^2}{5}$. If we assign to x any numerical value the function will have a corresponding value.

Let us assign to x the values 1, 2, 3, 4, &c., then the function $\frac{x^2}{5}$ will have the values ·2, ·8, 1·8, 3·2, &c., respectively.

The student will observe that as x increases from 1 to 2 the function is thereby increased from ·2 to ·8, and as x increases from 2 to 3 the function is increased from ·8 to 1·8, and as x increases from 3 to 4 the function is increased from 1·8 to 3·2.

Observe that as x increases by equal and finite increments the function does not increase by equal increments, but the increase of the function depends upon the assigned value of x, from which it is supposed to increase.

In the above function, as x increases the increment of the function is positive; but the increment of a function may be positive, negative, or zero, depending upon the *form* of the function and also upon the assigned value of x.

In the differential calculus we investigate the change which a function undergoes when the variable upon which it depends is increased by an indefinitely small quantity; and the practical utility of the subject depends upon the degree of rapidity with which the function varies when the variable upon which it depends varies from some assigned value.

In the example under consideration x is called the *independent variable*, and the function y the dependent variable; because the value of the function depends upon whatever value we assign to x.

ART. 3.—Representation of Functions and Differential Coefficients.

In Fig. 1, O P Q is a curve, whose equation is $y = \frac{1}{5} x^2$, showing the relation between the function y and the variable x.

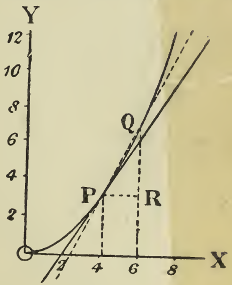


FIG. 1.

Distances along O X represent the variable x, and distances along O Y the function y.

At the point P, $x = 4$, and $y = \frac{16}{5}$.

" " Q, $x = 6$, " $y = \frac{36}{5}$.

Whilst x increases from 4 to 6 the function y is increased from $3\frac{1}{5}$ to $7\frac{1}{5}$, that is, if increment of x is 2 the increment of the function is 4; therefore

$$\frac{\text{Increment of } y}{\text{Increment of } x} = \frac{4}{2},$$

where the increment of x is finite.

Again, on referring to the curve, suppose P fixed, and let Q gradually move indefinitely near to P, the line P Q will turn about P; consequently

$$\frac{\text{Increment of } y}{\text{Increment of } x}$$

will continually vary until Q is indefinitely near to P, and when Q and P are indefinitely near the straight line drawn through them is a tangent to the curve O P Q, and for that position of the line through P Q, the increment of y, Q R, and the increment of x, P R, are indefinitely small. Denoting these values of Q R and P R by $d\,y$ and $d\,x$ respectively, we have $\frac{d\,y}{d\,x}$ = tangent of the inclination of the curve O P Q, at the point P, to the axis of X.

This ratio $\frac{dy}{dx}$, or increment of function, y, to increment of variable, x, when the increment of x is indefinitely small, is called the *differential coefficient* of the function with respect to the variable or *first derived function.* It is obvious that the value of this ratio depends upon the position of the point P on the curve.

Another way of expressing a differential coefficient is as follows.

Let $y = f(x)$, and suppose x to receive an indefinitely small increment, dx, and in consequence let the increment of y be dy; therefore

$$dy = f(x + dx) - f(x).$$

Hence

$$\frac{dy}{dx} = \frac{f(x + dx) - f(x)}{dx}.$$

A differential coefficient may also be illustrated in the following way.

Suppose a railway train to run from one town to another 50 miles distant, in 2 hours, its average velocity is 25 miles an hour; but during an hour its velocity at one time may be at the rate of 50 miles an hour or more, and at another time it may be 5 miles an hour, or 0, if it comes to rest. If we plot time t horizontally and distance s vertically, the distance being 0 when the time is 0, the slope of the curve at any point will represent the true velocity at that instant. For let $s_{,}$ denote the space passed over in $t_{,}$ seconds, and $s_{,,}$ the space passed over in $t_{,,}$ seconds, then $s_{,} = v\,t_{,}$ and $s_{,,} = v\,t_{,,}$ where v

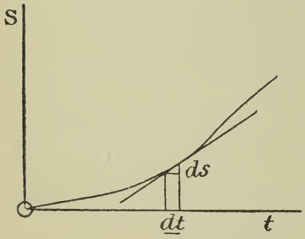


Fig. 2.

is the average velocity; therefore, by subtracting, we have

$$s_{,,} - s_{,} = v\,(t_{,,} - t_{,}),$$

or

$$v = \frac{s_{,,} - s_{,}}{t_{,,} - t_{,}}.$$

If we make the difference between $t_{,,}$ and $t_{,}$ indefinitely small, and consequently the difference between $s_{,,}$ and $s_{,}$ indefinitely small, we have

$$v = \frac{d\,s}{d\,t},$$

where $d\,s = s_{,,} - s_{,}$, and $d\,t = t_{,,} - t_{,}$; that is, the velocity at any instant is the rate at which distance increases as time increases, or it is the slope of the curve. Similarly, an acceleration a may be shown to be

$$a = \frac{d\,v}{d\,t},$$

that is,

$$a = \left(\frac{d}{d\,t}\right)\left(\frac{d\,s}{d\,t}\right) = \frac{d^2\,s}{d\,t^2}.$$

Hence an acceleration is the rate at which a *velocity* increases as time increases, or rate of change of slope.

Art. 4.—Differentiation

is the process of finding the *differential coefficient* of a function.

We proceed to illustrate this process by means of a few examples.

(1.) Let

$$y = \frac{x^2}{5}. \quad . \quad . \quad . \quad . \quad . \quad (a)$$

Suppose x to receive a finite increment which we shall denote by $\delta\,x$ (here $\delta\,x$ is equivalent to P R in Fig. 1), and in consequence let y become $y + \delta y$ (here δy is equivalent t QR in Fig. 1); therefore

$$y + \delta y = \frac{(x + \delta x)^2}{5} \quad . \quad . \quad . \quad . \quad . \quad (b)$$

Subtracting (a) from (b) we get

$$\delta y = \frac{2}{5} x \, \delta x + \frac{1}{5} (\delta x)^2.$$

Now suppose δx to get smaller and smaller until it becomes indefinitely small. (This is equivalent to the point Q moving along the curve in Fig. 1, until it approaches indefinitely near to P), and let this value be denoted by $d x$, and let the corresponding value of δy be denoted by $d y$.

Then $d y = \frac{2}{5} x \, d x +$ the square of an indefinitely smal quantity, which may be neglected.

$$\therefore \; d y = \frac{2}{5} x \, d x. \quad . \quad . \quad . \quad . \quad (c)$$

The student may here ask, why has the square of an indefinitely small term been neglected, when the term $\frac{2}{5} x \, d$ which is also indefinitely small, ha not been neglected? In answer to this, let us consider the portion of the curve between P and Q, when the distance between these points is, say, one-millionth of an inch, and let this distance be magnified one million times.

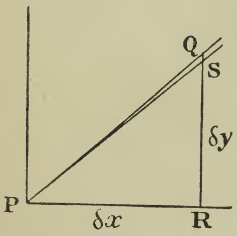


Fig. 3.

In Fig. 3, P S is a portion of the tangent to the curve at the point P, R Q representing the true value of δy, and R S representing its value when the aforesaid term is neglected. Now suppose the distance P Q to become indefinitely small, it is obvious that the distance Q S will in consequence become indefinitely small in comparison with R S, that is, $d\,y$, which is also indefinitely small.

In equation (c) $d\,y$ or its equivalent $\frac{2}{5}\,x\,dx$ is called the *differential* of the function, and is always an indefinitely small quantity, since one of its factors, namely, the increment of the independent variable, is infinitely small.

Divide both sides of equation (c) by $d\,x$.

$$\therefore \quad \frac{d\,y}{d\,x} = \frac{2}{5}\,x. \quad . \quad . \quad . \quad . \quad . \; (d)$$

That is, $\frac{2}{5}\,x$ is the *differential coefficient* of $\frac{x^2}{5}$, and its value depends on whatever value we assign to x.

The student should remember the distinction between *differential* and *differential coefficient*, the former being always infinitely small; the latter may have any value, since it represents the tangent of an angle.

If we substitute 4 for x, in equation (d), we get

$$\frac{d\,y}{d\,x} = \frac{2}{5} \times 4 = \frac{8}{5}.$$

On referring to Fig. 1, we find that the tangent to the curve, at the point P, whose abscissa is 4, makes an angle with the axis of X, whose tangent is $\frac{8}{5}$.

(2.) Let

$$f(x), \text{ or } y = 2\,x + 7, \quad . \quad . \quad . \quad . \; (a)$$

and suppose x increased by an indefinitely small quantity $d\,x$, and let the corresponding change in the function, or y, be denoted by $d\,y$. Then

$$y + d\,y = 2\,(x + d\,x) + 7. \quad . \quad . \quad . \; (b)$$

Subtracting (a) from (b) we have

$$\therefore \quad d\,y = 2\,(x + d\,x) + 7 - 2\,x - 7 = 2\,d\,x.$$

$$\therefore \quad \frac{d\,y}{d\,x} = 2. = \frac{2\,d\,x}{d\,\cancel{x}}$$

The equation $y = 2x + 7$ represents a straight line inclined to the axis of X, at an angle whose tangent is 2.

(3.)
$$y = x^3. \quad . \quad . \quad . \quad . \quad . \quad (a)$$

Suppose x to receive an indefinitely small increment dx, and in consequence let y become $y + dy$, therefore (a) becomes

$$y + dy = (x + dx)^3. \quad . \quad . \quad . \quad (b)$$

Subtracting (a) from (b) we have

$$dy = (x + dx)^3 - x^3;$$

therefore

$$dy = 3x^2 dx + 3x(dx)^2 + (dx)^3,$$

and on dividing by dx we get

$$\frac{dy}{dx} = 3x^2 + 3x\,dx + (dx)^2.$$

The terms $3x\,dx$ and $(dx)^2$ are indefinitely small compared with $3x^2$, and may, therefore, be neglected; therefore if

$$y = x^3, \quad \frac{dy}{dx} = 3x^2.$$

(4.) If $y = x^n$, $\therefore dy = (x + dx)^n - x^n$, and on expanding by the Binomial Theorem, we get

$$dy = n x^{n-1} dx + \frac{n(n-1)}{\underline{|2}} x^{n-2} (dx)^2 + \&c.$$

$$\therefore \frac{dy}{dx} = n x^{n-1} + \text{terms involving } dx \text{ and higher powers}$$

of it which may be neglected.

$$\therefore \frac{dy}{dx} = n x^{n-1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A)$$

This is true for all values of n, since

$$(x + dx)^n = x^n + n x^{n-1} dx + \&c.$$

for all values of n.

Therefore we deduce the following rule:

Rule I.—The differential coefficient of a power of x is obtained by multiplying the power of x by the original index, and diminishing the index by unity.

We add a few examples to illustrate this rule.

(1.) If $y = x^9$, $\quad \dfrac{d\,y}{d\,x} = 9\,x^8$.

(2.) „ $y = x^{-19}$, $\quad \dfrac{d\,y}{d\,x} = -\,19\,x^{-20}$

(3.) „ $y = x^{\frac{3}{4}}$, $\quad \dfrac{d\,y}{d\,x} = \frac{3}{4}\,x^{-\frac{1}{4}}$.

(4.) „ $y = x^{-\frac{4}{3}}$, $\quad \dfrac{d\,y}{d\,x} = -\,\frac{4}{3}\,x^{-\frac{7}{3}}$.

ART. 5.—Differential Coefficient of the Algebraic Sum of any Number of Functions.

Let us take, for example, the function,

(5.) $$y = x^4 + 4\,x^3 + 6\,x^2 + 4\,x. \quad . \quad . \quad . \quad (a)$$

We may consider each term on the right-hand side as being a separate function of x.

Suppose x to become $x + d\,x$, $d\,x$ being indefinitely small, and in consequence let y become $y + d\,y$; therefore (a) becomes

$$y + d\,y = (x + d\,x)^4 + 4\,(x + d\,x)^3 + 6\,(x + d\,x)^2 + 4\,(x + d\,x). \quad . \quad . \quad . \quad . \quad . \quad (b)$$

Subtracting (a) from (b) we have

$$d\,y = \{(x + d\,x)^4 - x^4\} + 4\,\{(x + d\,x)^3 - x^3\} + 6\,\{(x + d\,x)^2 - x^2\} + 4\,\{(x + d\,x) - x\}.$$

On expanding the terms on the right-hand side and collecting, we have

$$d\,y = (4\,x^3 + 12\,x^2 + 12\,x + 4)\,d\,x + \text{terms involving } (d\,x)^2 \text{ and higher powers of } (d\,x).$$

Divide both sides of this equation by $d\,x$, and we have

$\frac{d\,y}{d\,x} = 4\,x^3 + 12\,x^2 + 12\,x + 4 +$ terms involving $d\,x$, and higher powers of $d\,x$, which may be neglected, since they are infinitely small in comparison with the expression

$$4\,x^3 + 12\,x^2 + 12\,x + 4,$$

$$\therefore\ \frac{d\,y}{d\,x} = 4\,x^3 + 12\,x^2 + 12\,x + 4,$$

But by Rule I.,

$4\,x^3$ is the differential coefficient of x^4
and $12\,x^2$ „ „ $4\,x^3$
„ $12\,x$ „ „ $6\,x^2$
„ 4 „ „ $4\,x$.

(6.) Again, let

$$y = z + v - u + \&c. \quad . \quad . \quad . \quad (a)$$

where z, v, u, &c., are functions of x, and let these functions become $z_{,}$, $v_{,}$, $u_{,}$, &c. respectively, when x is increased by an infinitely small quantity $d\,x$, and in consequence suppose y changed to $y_{,}$,

$$\therefore\ y_{,} = z_{,} + v_{,} - u_{,} + \&c. \quad . \quad . \quad . \quad (b)$$

Subtracting (a) from (b) we have

$$y_{,} - y = (z_{,} - z) + (v_{,} - v) - (u_{,} - u) + \&c.$$

Now $y_{,} - y$ represents the increment of the algebraic sum of the functions, z, v, u, &c., and is therefore the differential of y, or $d\,y$, similarly $z_{,} - z$, is $d\,z$, &c.

$$\therefore\ d\,y = d\,z + d\,v - d\,u + \&c.$$

And since z, v, u, &c., are functions of x, dividing both s les of the above equation by $d\,x$, we have

$$\frac{d\,y}{d\,x} = \frac{d\,z}{d\,x} + \frac{d\,v}{d\,x} - \frac{d\,u}{d\,x} + \&c. \quad . \quad . \quad . \quad (B)$$

Hence we arrive at the following rule:

Rule II. The differential coefficient or first de- i ved function of the algebraic sum of any number of nctions is the algebraic sum of the differential co- ficients of the functions.

ART. 6.—Differential Coefficient of the Product of Two or more Functions.

(7.) Let

$$y = u\,v \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

vhere u and v are functions of x, and suppose x to increase $x + d\,x$, and in consequence let (a) become

$$y + d\,y = (u + d\,u)\,(v + d\,v). \quad . \quad . \quad (b)$$

'herefore

$$d\,y = (u + d\,u)\,(v + d\,v) - u\,v.$$

'hat is

$$d\,y = u\,d\,v + v\,d\,u + d\,u\,d\,v.$$

The term, $d\,u\,d\,v$, may be neglected, since it is infinitely nall compared with the other terms.

Therefore we have $d\,y = u\,d\,v + v\,d\,u$, and on dividing by $d\,x$ we have

$$\frac{d\,y}{d\,x} = u\,\frac{d\,v}{d\,x} + v\,\frac{d\,u}{d\,x}. \quad . \quad . \quad . \quad . \quad (C)$$

That is, the differential coefficient of the product of two functions is obtained by multiplying each function by the differential coefficient of the other function, and taking the sum of the products.

Let us take three factors.

Suppose

$$y = u\,v\,z \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

where u, v, and z are functions of x, and in consequence of being increased by an infinitely small quantity, $d\,x$, equation (a) become

$$y + d\,y = (u + d\,u)\,(v + d\,v)\,(z + d\,z) \quad . \quad . \quad (b)$$

$$\therefore \quad d\,y = (u + d\,u)\,(v + d\,v)\,(z + d\,z) - u\,v\,z.$$

$\therefore$ $d\,y = u\,v\,d\,z + u\,z\,d\,v + v\,z\,d\,u +$ terms which are infinitely small in comparison with any one of these, and may therefore be neglected, since they contain the product of two or more differentials. Now the product of two differentials is infinitely small in comparison with either of its factors since it is an infinitely small fraction of either factor;

$$\therefore\ d\,y = u\,v\,d\,z + u\,z\,d\,v + v\,z\,d\,u.$$

Dividing both sides of this equation by $d\,x$, we have

$$\frac{d\,y}{d\,x} = u\,v\,\frac{d\,z}{d\,x} + u\,z\,\frac{d\,v}{d\,x} + v\,z\,\frac{d\,u}{d\,x}. \quad . \quad . \quad (\mathrm{D})$$

From equation (D) we deduce the following rule:

Rule III. The differential coefficient of the product of n functions of x is the sum of all such terms as the product of $n - 1$ factors into the differential coefficient of the remaining factor.

Dividing the left-hand side of equation (D) by y, and the right-hand side by $u\,v\,z$, we have

$$\frac{1}{y}\,\frac{d\,y}{d\,x} = \frac{1}{z}\,\frac{d\,z}{d\,x} + \frac{1}{v}\,\frac{d\,v}{d\,x} + \frac{1}{u}\,\frac{d\,u}{d\,x}. \quad . \quad . \quad (\mathrm{E})$$

We add a few examples to illustrate the above rules.

(8.) Let

$$y = (x^2 + 3)\,(x^3 + 4),$$

where x^2+3 and x^3+4 are considered to be separate functions of x, and are equivalent to u and v, respectively, in (C).

Using formula C we have

$$\frac{dy}{dx} = (x^2+3)\frac{d(x^3+4)}{dx} + (x^3+4)\frac{d(x^2+3)}{dx}.$$

Now

$$\frac{d(x^3+4)}{dx} = 3x^2,$$

and

$$\frac{d(x^2+3)}{dx} = 2x,$$

therefore

$$\frac{dy}{dx} = 3x^2(x^2+3) + 2x(x^3+4) = 5x^4 + 9x^2 + 8x.$$

Otherwise we have, on multiplying the factors together,

$$y = x^5 + 3x^3 + 4x^2 + 12,$$

therefore, by Rule II. we have

$$\frac{dy}{dx} = 5x^4 + 9x^2 + 8x.$$

(9.) $$y = (x^2+a^2)(x^2+b^2)(x^2+c^2).$$

By formula D, we have

$$\frac{dy}{dx} = (x^2+a^2)(x^2+b^2)\frac{d(x^2+c^2)}{dx}$$

$$+ (x^2+a^2)(x^2+c^2)\frac{d(x^2+b^2)}{dx}$$

$$+ (x^2+b^2)(x^2+c^2)\frac{d(x^2+a^2)}{dx}.$$

But

$$\frac{d(x^2+a^2)}{dx} = 2x = \frac{d(x^2+b^2)}{dx} = \frac{d(x^2+c^2)}{dx},$$

therefore

$$\frac{dy}{dx} = 6x^5 + 4(a^2 + b^2 + c^2)x^3 + 2(a^2b^2 + a^2c^2 + b^2c^2).$$

Otherwise we have

$$y = x^6 + (a^2 + b^2 + c^2)x^4 + (a^2b^2 + a^2c^2 + b^2c^2)x^2 + a^2b^2c^2,$$

therefore by B,

$$\frac{dy}{dx} = 6x^5 + 4(a^2 + b^2 + c^2)x^3 + 2(a^2b^2 + a^2c^2 + b^2c^2)x.$$

It will be noticed, in the preceding examples, that the differential coefficient of the constant term is always cipher, since it receives no increment, and in the process of differentiation it is always subtracted from itself.

*(10.) $y = x^3 \sin x \sin^{-1} x.$

By formula (D) we have

$$\frac{dy}{dx} = x^3 \sin x \frac{d(\sin^{-1} x)}{dx} + x^3 \sin^{-1} x \frac{d(\sin x)}{dx}$$
$$+ \sin x \sin^{-1} x \frac{d(x^3)}{dx},$$

therefore

$$\frac{dy}{dx} = \frac{x^3 \sin x}{\sqrt{1 - x^2}} + x^3 \sin^{-1} x \cos x + 3x^2 \sin x \sin^{-1} x.$$

ART. 7.—Differential Coefficient of a Quotient.

(11.) Let

$$y = \frac{z}{v},$$

where z and v are functions of x.

Clearing of fractions, we have

$$z = vy,$$

therefore by (C),

$$\frac{dz}{dx} = v\frac{dy}{dx} + y\frac{dv}{dx}.$$

---

* See differential coefficients of $\sin x$ and $\sin^{-1} x$.

Transposing, and substituting $\frac{z}{v}$ for y, we have

$$v\frac{dy}{dx} = \frac{dz}{dx} - \frac{z}{v}\frac{dv}{dx},$$

and on dividing by v, we have

$$\frac{dy}{dx} = \frac{v\frac{dz}{dx} - z\frac{dv}{dx}}{v^2}. \quad . \quad . \quad . \quad . \quad (F)$$

From this result we deduce the following rule:

Rule IV.—The differential coefficient of a quotient is obtained by multiplying the denominator into the differential coefficient of the numerator, and the numerator into the differential coefficient of the denominator; subtract the latter product from the former, and divide by the square of the denominator.

(12.)
$$y = \frac{x^3 + 2}{x^2},$$

$$\therefore \quad \frac{dy}{dx} = \frac{x^2 \times 3x^2 - (x^3 + 2) \times 2x}{x^4} = \frac{x^3 - 4}{x^3}.$$

(13.)
$$y = \frac{(x + a)(x + b)}{(x + c)},$$

$$\therefore \quad \frac{dy}{dx} = \frac{(x + c)(2x + a + b) - (x + a)(x + b)}{(x + c)^2}$$

$$= \frac{x^2 + 2cx + ac + bc - ab}{(x + c)^2}.$$

Sometimes it is more convenient to express the equation $y = \frac{z}{v}$ thus: $y = zv^{-1}$, and treat it as a product.

By equation (C),

$$\frac{dy}{dx} = z\frac{d(v^{-1})}{dx} + v^{-1}\frac{dz}{dx}.$$

If

$$y = \frac{x^3 + 2}{x^2} = (x^3 + 2)x^{-2},$$

$$\therefore \quad \frac{dy}{dx} = (x^3 + 2) \times - 2x^{-3} + x^{-2} \times 3x^2$$

$$= -2 - 4x^{-3} + 3$$

$$= 1 - 4x^{-3},$$

$$\therefore \quad \frac{dy}{dx} = \frac{x^3 - 4}{x^3}.$$

ART. 8.—Differentiation of a Function of a Function.

Let $y = f(z)$, and $z = \mathrm{F}(x)$.

Suppose x to receive a finite increment, which we shall denote by δx, and in consequence let the corresponding increments of z and y be denoted by δz and δy respectively.

By the property of fractions we have

$$\frac{\delta y}{\delta x} = \frac{\delta y}{\delta z} \times \frac{\delta z}{\delta x},$$

and this is true for finite values of the increments, however small; therefore it must hold in the limit when δx becomes indefinitely small. Therefore

$$\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx}. \quad . \quad . \quad . \quad . \quad \text{(G)}$$

(14.) $y = (a + cx^4)^2$.

Let $z = a + cx^4$, and the above equation becomes

$$y = z^2,$$

$$\therefore \quad \frac{dy}{dz} = 2z.$$

And

$$\frac{dz}{dx} = 4cx^3.$$

By equation (G),

$$\frac{dy}{dx} = 2z \times 4cx^3.$$

Substituting for z, we have

$$\frac{dy}{dx} = 8\,c\,x^3\,(a + c\,x^4).$$

Also it may be shown as above, that if $y = f(z)$, $z = \phi(v)$, and $v = \psi(x)$, then

$$\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dv} \times \frac{dv}{dx}. \quad . \quad . \quad . \quad \text{(H)}$$

(15.) Let

$$y = (x^3 - 6\,x^2 + 4\,x + 1)^4.$$

Here the expression inside the bracket is a function of x, which we shall denote by z,

$$\therefore \quad y = z^4,$$

Therefore y is a function of z, and z is a function of x, or $y = f(z)$, $z = \phi(x)$.

By equation (A),

$$\frac{dy}{dz} = 4\,z^3,$$

nd

$$\frac{dz}{dx} = 3\,x^2 - 12\,x + 4,$$

y equation (B); since

$$z = x^3 - 6\,x^2 + 4\,x + 1,$$

$$\therefore \quad \frac{dy}{dx} = 4\,(x^3 - 6\,x^2 + 4\,x + 1)^3\,(3\,x^2 - 12\,x + 4).$$

(16.) $$y = (a^2 + x^2)^{\frac{1}{2}}.$$

et $$a^2 + x^2 = z,$$

$$\therefore \quad y = z^{\frac{1}{2}},$$

$$\therefore \quad \frac{dy}{dz} = \frac{1}{2}\,z^{-\frac{1}{2}} = \frac{1}{2\,z^{\frac{1}{2}}}$$

nd

$$\frac{dz}{dx} = 2\,x,$$

and by (G),

$$\frac{dy}{dx} = \frac{dy}{dz}\,\frac{dz}{dx} = \frac{1}{2\,z^{\frac{1}{2}}} \times 2\,x,$$

$$\therefore \quad \frac{dy}{dx} = \frac{x}{(a^2 + x^2)^{\frac{1}{2}}}$$

(17.) Given $v = x^3$, $z = v^2 + 3\,v$, and $y = z^{\frac{1}{3}}$; find $\frac{dy}{dx}$.

Here

$$\frac{dv}{dx} = 3\,x^2, \ \frac{dz}{dv} = 2\,v + 3,$$

and

$$\frac{dy}{dz} = \frac{1}{3}\,z^{-\frac{2}{3}},$$

therefore by (H) we have

$$\frac{dy}{dx} = \frac{1}{3}\,z^{-\frac{2}{3}}\,(2\,v + 3) \times 3\,x^2,$$

that is,

$$\frac{dy}{dx} = (x^6 + 3\,x^3)^{-\frac{2}{3}}(2\,x^3 + 3)\,x^2.$$

ART. 9.—Differentiation of the Trigonometrical Functions.

Lemma.—The limiting values of $\frac{\sin x}{x}$ and $\frac{\tan x}{\sin x}$ is unity. Let A C be a very small arc of a circle whose radius O A is unity; and let A B be a tangent at the point A, and C D the perpendicular from C, on O A; then C D, C A and A B represent the sine, angle in radians and tangent of the angle C O A, or x, respectively.

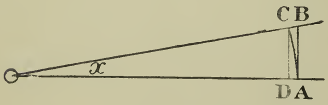


FIG. 4.

It is evident from the figure that $\sin x$, x and $\tan x$ are in ascending order of magnitude.

We have,

$$\sin x < x < \tan x,$$

dividing by $\sin x$,

$$1 < \frac{x}{\sin x} < \frac{\tan x}{\sin x}.$$

But

$$\frac{\tan x}{\sin x} = \frac{1}{\cos x},$$

and when x is diminished indefinitely,

$$\cos x = 1;$$

therefore the limiting value of $\frac{\tan x}{\sin x}$ is unity. And since $\frac{x}{\sin x}$ lies, in magnitude, between 1 and $\frac{\tan x}{\sin x}$, therefore its limiting value is also unity, and hence its reciprocal $\frac{\sin x}{x}$ is unity, when x is indefinitely small.

(18.) Let $y = \sin x$, and suppose x to increase by an infinitely small quantity $d\,x$, and in consequence let y become $y + d\,y$.

$$\therefore \quad y + d\,y = \sin(x + d\,x),$$

$$\therefore \quad d\,y = \sin(x + d\,x) - \sin x$$

$$= 2\cos\left(x + \frac{d\,x}{2}\right)\sin\frac{d\,x}{2},$$

$$\therefore \quad \frac{d\,y}{d\,x} = \cos\left(x + \frac{d\,x}{2}\right)\frac{\sin\frac{d\,x}{2}}{\frac{d\,x}{2}}.$$

But

$$\frac{\sin\frac{d\,x}{2}}{\frac{d\,x}{2}} = 1,$$

and $\frac{dx}{2}$ may be neglected in comparison with x in $\cos\left(x + \frac{dx}{2}\right)$.

$$\therefore \quad \frac{dy}{dx} = \cos x.$$

(19.) $$y = \cos x,$$

$$\therefore \quad y + dy = \cos(x + dx),$$

$$\therefore \quad dy = \cos(x + dx) - \cos x$$

$$= -2 \sin\left(x + \frac{dx}{2}\right) \sin \frac{dx}{2}.$$

$$\therefore \quad \frac{dy}{dx} = -\sin\left(x + \frac{dx}{2}\right) \frac{\sin \frac{dx}{2}}{\frac{dx}{2}}.$$

$$\frac{dy}{dx} = -\sin x.$$

(20.) $$y = \tan x,$$

$$y + dy = \tan(x + dx),$$

$$\therefore \quad dy = \tan(x + dx) - \tan x,$$

$$= \frac{\sin(x + dx)\cos x - \cos(x + dx)\sin x}{\cos(x + dx)\cos x}$$

$$= \frac{\sin dx}{\cos(x + dx)\cos x};$$

therefore

$$\frac{dy}{dx} = \frac{\frac{\sin dx}{dx}}{\cos(x + dx)\cos x}.$$

Hence

$$\frac{dy}{dx} = \frac{1}{\cos^2 x} = \sec^2 x.$$

Otherwise we have

$$y = \tan x = \frac{\sin x}{\cos x},$$

and by (F),

$$\frac{d\,y}{d\,x} = \frac{\cos x \dfrac{d\,(\sin x)}{d\,x} - \sin x \dfrac{d\,(\cos x)}{d\,x}}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

(20*a*.) $$y = \cot x = \frac{\cos x}{\sin x}.$$

By (F) we have

$$\frac{d\,y}{d\,x} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\operatorname{cosec}^2 x.$$

(21.) $$y = \sec x = \frac{1}{\cos x};$$

therefore

$$d\,y = \frac{1}{\cos(x + d\,x)} - \frac{1}{\cos x}$$

$$= \frac{\cos x - \cos(x + d\,x)}{\cos(x + d\,x)\cos x};$$

therefore

$$\frac{d\,y}{d\,x} = \frac{\sin\left(x + \dfrac{d\,x}{2}\right) \dfrac{\sin \dfrac{d\,x}{2}}{\dfrac{d\,x}{2}}}{\cos(x + d\,x)\cos x}.$$

Hence

$$\frac{d\,y}{d\,x} = \frac{\sin x}{\cos^2 x} = \sec x \tan x.$$

(22) $$y = \operatorname{cosec} x = \frac{1}{\sin x},$$

therefore

$$d\,y = \frac{1}{\sin(x + d\,x)} - \frac{1}{\sin x};$$

$$\therefore\ d\,y = \frac{\sin x - \sin(x + d\,x)}{\sin x \sin(x + d\,x)} = \frac{-2\cos\left(x + \frac{d\,x}{2}\right)\sin\frac{d\,x}{2}}{\sin x \sin(x + d\,x)};$$

$$\therefore\ \frac{d\,y}{d\,x} = \frac{-\cos\left(x + \frac{d\,x}{2}\right)\dfrac{\sin\frac{d\,x}{2}}{\frac{d\,x}{2}}}{\sin x \sin(x + d\,x)}$$

$$= -\frac{\cos x}{\sin^2 x};$$

$$\therefore\ \frac{d\,y}{d\,x} = -\operatorname{cosec} x \cot x.$$

(23.) $y = \operatorname{versin} x = 1 - \cos x;$

therefore by example (19),

$$\frac{d\,y}{d\,x} = \sin x.$$

(24.) $y = \operatorname{coversin} x = 1 - \sin x,$

therefore by example (17),

$$\frac{d\,y}{d\,x} = -\cos x.$$

In the eight examples given above, it will be noticed that the differential coefficients of sin x, tan x, sec x and versin x, have the same sign as the functions themselves; since these functions increase as x increases. And cos x, cot x, cosec x, and coversin x, have the opposite signs to that of the functions, since these functions diminish as x increases.

ART 10.—Differential Coefficients of the Inverse Trigonometrical Functions.

Suppose y to be some function of x,

$$y = \mathrm{F}(x). \quad . \quad . \quad . \quad . \quad . \quad (a)$$

If this equation admits of being expressed in the form

$$x = \phi(y) \quad . \quad . \quad . \quad . \quad . \quad (b)$$

then

$$\frac{dy}{dx} \times \frac{dx}{dy} = 1.$$

For if in equations (a) and (b), x and y be increased by finite increments δx and δy respectively, then, by the property of fractions,

$$\frac{\delta x}{\delta y} \times \frac{\delta y}{\delta x} = 1,$$

for all finite values of the increments, however small; therefore it must hold in the limit when δy and δx become indefinitely small.

$$\therefore \quad \frac{dy}{dx} \times \frac{dx}{dy} = 1,$$

or

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{I})$$

(25.) $y = \sin^{-1} x$.

Here y is a function of x, and it may be easily expressed in the form $x = \phi(y)$, that is,

$$x = \sin y;$$

therefore by (18),

$$\frac{dx}{dy} = \cos y = \pm \sqrt{(1 - x^2)};$$

since $\cos y = \sqrt{1 - \sin^2 y}$.

therefore by (I),

$$\frac{dy}{dx} = \pm \frac{1}{\sqrt{1 - x^2}}.$$

The proper sign depends upon the value we assign to y; since, for a particular value of x, we have more than one value of y.

If we assign to y its least value for any value of x between 1 and 0, then we must take the positive sign, since the function increases as x increases.

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sqrt{(1 - x^2)}}.$$

(26.) $$y = \cos^{-1} x,$$

$$\therefore \quad x = \cos y,$$

$$\therefore \quad \frac{dx}{dy} = -\sin y = -\sqrt{(1 - x^2)},$$

therefore by (I),

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}}.$$

(27.) $$y = \tan^{-1} x,$$

$$\therefore \quad x = \tan y,$$

therefore by (19),

$$\frac{dx}{dy} = \sec^2 y = 1 + x^2,$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{1 + x^2}.$$

(28.) $$y = \cot^{-1} x,$$

$$\therefore \quad x = \cot y,$$

therefore by (20),

$$\frac{dx}{dy} = -\operatorname{cosec}^2 y = -(1 + x^2),$$

$$\therefore \quad \frac{dy}{dx} = -\frac{1}{1 + x^2}.$$

(29.)
$$y = \sec^{-1} x,$$
$$\therefore \quad x = \sec y,$$

therefore by (21),

$$\frac{dx}{dy} = \sec y \tan y = x\sqrt{(x^2 - 1)},$$
$$\therefore \quad \frac{dy}{dx} = \frac{1}{x\sqrt{(x^2 - 1)}}.$$

(30.)
$$y = \operatorname{cosec}^{-1} x,$$
$$\therefore \quad x = \operatorname{cosec} y,$$
$$\therefore \quad \frac{dx}{dy} = -\operatorname{cosec} y \cot y = -x\sqrt{(x^2 - 1)},$$
$$\therefore \quad \frac{dy}{dx} = -\frac{1}{x\sqrt{(x^2 - 1)}}.$$

(31.)
$$y = \operatorname{versin}^{-1} x,$$
$$\therefore \quad x = \operatorname{versin} y = 1 - \cos y,$$

therefore

$$\frac{dx}{dy} = \sin y = \sqrt{2x - x^2}.$$

Hence

$$\frac{dy}{dx} = \frac{1}{\sqrt{2x - x^2}}.$$

(32.)
$$y = \operatorname{coversin}^{-1} x,$$
$$\therefore \quad x = \operatorname{coversin} y = 1 - \sin y,$$

and

$$\frac{dx}{dy} = -\cos y = -\sqrt{2x - x^2},$$

therefore

$$\frac{dy}{dx} = -\frac{1}{\sqrt{2x - x^2}}.$$

ART. 11.—Differentiation of Logarithmic and Exponential Functions.

Here we shall assume the Exponential Theorem, wh we shall afterwards prove in the chapter on the expansion functions, namely, that

$$a^x = 1 + x \log_e a + \frac{x^2}{\lfloor\underline{2}} (\log_e a)^2 + \frac{x^3}{\lfloor\underline{3}} (\log_e a)^3$$

$$+ \frac{x^4}{\lfloor\underline{4}} (\log_e a)^4 + \frac{x^5}{\lfloor\underline{5}} (\log_e a)^5 + \&c.,$$

where e is the base of the Napierian logarithms.

(33.) Let
$$y = \log_a x,$$
therefore
$$x = a^y,$$
that is,
$$x = 1 + y \log_e a + \frac{y^2}{\lfloor\underline{2}} (\log_e a)^2 + \frac{y^3}{\lfloor\underline{3}} (\log_e a)^3 + \&c.,$$
therefore
$$\frac{dx}{dy} = \log_e a + y (\log_e a)^2 + \frac{y^2}{\lfloor\underline{2}} (\log_e a)^3 + \&c.,$$

and on dividing both sides by $\log_e a$, we have

$$\frac{1}{\log_e a} \frac{dx}{dy} = 1 + y \log_e a + \frac{y^2}{\lfloor\underline{2}} (\log_e a)^2 + \frac{y^3}{\lfloor\underline{3}} (\log_e a)^3 + \&$$

therefore
$$\frac{1}{\log_e a} \frac{dx}{dy} = x.$$

Hence
$$\frac{dx}{dy} = x \log_e a = \frac{x}{\log_a e},$$
therefore
$$\frac{dy}{dx} = \frac{\log_a e}{x}.$$

therwise,—If $y = \log_a x$

ave

$$dy = \log_a (x + dx) - \log_a x = \log_a \left(1 + \frac{dx}{x}\right),$$

refore

$$\frac{dy}{dx} = \frac{1}{dx} \log_a \left(1 + \frac{dx}{x}\right). \quad . \quad . \quad . \quad (a)$$

Let $dx = \alpha x$ where α is indefinitely small and (a) becomes

$$\frac{dy}{dx} = \frac{1}{\alpha x} \log_a (1 + \alpha) = \frac{1}{x} \log_a (1 + \alpha)^{\frac{1}{\alpha}}.$$

Expanding $(1 + \alpha)^{\frac{1}{\alpha}}$ by the Binomial Theorem, we have

$$(1 + \alpha)^{\frac{1}{\alpha}} = 1 + \frac{1}{\alpha} \times \alpha + \frac{\frac{1}{\alpha}\left(\frac{1}{\alpha} - 1\right)}{\underline{|2}} \alpha^2$$

$$+ \frac{\frac{1}{\alpha}\left(\frac{1}{\alpha} - 1\right)\left(\frac{1}{\alpha} - 2\right)}{\underline{|3}} \alpha^3 + \&c.$$

$$= 1 + 1 + \frac{1 - \alpha}{\underline{|2}} + \frac{(1 - \alpha)(1 - 2\alpha)}{\underline{|3}} + \&c.;$$

since α is indefinitely small we may neglect it, there-

$$(1 + \alpha)^{\frac{1}{\alpha}} = 1 + 1 + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \frac{1}{\underline{|4}} + \frac{1}{\underline{|5}} + \&c.$$

e the base of the Napierian logarithms, the numerical lue of which is 2·71828 . . . .

Hence

$$\frac{dy}{dx} = \frac{1}{x} \log_a e.$$

If $a = e$ we have

$$\frac{d\,y}{d\,x} = \frac{1}{x}.$$

(34.) $$y = n^x,$$

then

$$\log y = x \log n;$$

Taking the differentials, we have

$$d\,(\log y) = d\,(x \log n),$$

that is

$$\frac{d\,y}{y} = d\,x \log n,$$

therefore

$$\frac{1}{y}\frac{d\,y}{d\,x} = \log n,$$

hence,

$$\frac{d\,y}{d\,x} = y \log n = n^x \log n.$$

If $n = e$ in the above example then $\log n$ becomes $\log_e e = 1$, therefore if

$$y = e^x$$

then

$$\frac{d\,y}{d\,x} = e^x.$$

This, or any multiple of it, is the only function whose differential coefficient, or first derived function, is the same as the primitive function.

We shall now work out a few examples on the preceding rules for finding the differential coefficients of functions.

(35.) $$y = a^{\sin x},$$

$$\log y = \sin x \log a,$$

therefore

$$\frac{1}{y}\;\frac{d\,y}{d\,x} = \cos x \log a,$$

is,

$$\frac{dy}{dx} = y \cos x \log a,$$

refore

$$\frac{dy}{dx} = a^{\sin x} \cos x \log a.$$

(36.) $$y = \sin n x^2 = \sin z, \text{ where } z = n x^2,$$

erefore

$$\frac{dy}{dz} = \cos z, \text{ and } \frac{dz}{dx} = 2 n x.$$

Hence by (G) we have

$$\frac{dy}{dx} = 2 n x \cos n x^2.$$

(37.) $$y = \log (1 + x) = \log z,$$

here

$$z = 1 + x,$$

$$\therefore \quad \frac{dy}{dz} = \frac{1}{z} = \frac{1}{1+x}, \text{ and } \frac{dz}{dx} = 1.$$

Therefore by (G),

$$\frac{dy}{dx} = \frac{1}{1+x}.$$

Similarly, if

$$y = \log (1 - x),$$

then

$$\frac{dy}{dx} = -\frac{1}{1-x}.$$

(38.) $$y = \sqrt{a^2 + x^2} = z^{\frac{1}{2}},$$

where

$$z = a^2 + x^2,$$

$$\frac{dy}{dz} = \frac{1}{2} z^{-\frac{1}{2}} = \frac{1}{2} \frac{1}{\sqrt{a^2 + x^2}}, \quad \frac{dz}{dx} = 2x,$$

therefore by (G),

$$\frac{dy}{dx} = \frac{x}{\sqrt{(a^2+x^2)}}.$$

(39.) $$y = \log (a^2 + x^2) = \log z,$$

where

$$z = a^2 + x^2,$$

$$\therefore \quad \frac{dy}{dz} = \frac{1}{z} = \frac{1}{a^2 + x^2}, \quad \frac{dz}{dx} = 2x,$$

$$\therefore \quad \frac{dy}{dx} = \frac{2x}{a^2 + x^2}.$$

(40.) $$y = \log (x^2 + 3x + 1)^3,$$

$$\therefore \quad y = 3 \log (x^2 + 3x + 1),$$

$$\therefore \quad y = 3 \log z, \text{ where } z = x^2 + 3x + 1,$$

hence,

$$\frac{dy}{dz} = \frac{3}{z}, \quad \frac{dz}{dx} = 2x + 3,$$

$$\therefore \quad \frac{dy}{dx} = \frac{3(2x + 3)}{x^2 + 3x + 1}.$$

(41.) $$y = \sqrt{(x + 1)} \sqrt[3]{(x - 3)} \sqrt[5]{(x^2 - 4)^3},$$

$$y = (x + 1)^{\frac{1}{2}} (x - 3)^{\frac{1}{3}} (x^2 - 4)^{\frac{3}{5}},$$

$$\log y = \tfrac{1}{2} \log (x + 1) + \tfrac{1}{3} \log (x - 3) + \tfrac{3}{5} \log (x^2 - 4),$$

$$\therefore \quad \frac{1}{y} \frac{dy}{dx} = \frac{1}{2(x + 1)} + \frac{1}{3(x - 3)} + \frac{3}{5} \times \frac{2x}{x^2 - 4}.$$

Hence,

$$\frac{dy}{dx} = \sqrt{(x + 1)} \sqrt[3]{(x - 3)} \sqrt[5]{(x^2 - 4)^3}$$

$$\times \left\{\frac{1}{2(x + 1)} + \frac{1}{3(x - 3)} + \frac{6x}{5(x^2 - 4)}\right\}.$$

(42.) $$y = \log \left\{x + \sqrt{(a^2 + x^2)}\right\},$$

or

$$y = \log z, \text{ where } z = x + \sqrt{(a^2 + x^2)}.$$

We have

$$\frac{dy}{dz} = \frac{1}{z} = \frac{1}{x + \sqrt{(a^2 + x^2)}},$$

nd

$$\frac{dz}{dx} = 1 + \frac{x}{\sqrt{(a^2+x^2)}} = \frac{x + \sqrt{(a^2+x^2)}}{\sqrt{a^2+x^2}},$$

herefore by formula (G),

$$\frac{dy}{dx} = \frac{1}{\sqrt{(a^2+x^2)}}.$$

(43.) $$y = \sqrt{1 + e^2 \sin^2 x} = z^{\frac{1}{2}},$$

where

$$z = 1 + e^2 \sin^2 x,$$

$$\frac{dy}{dz} = \frac{1}{2\sqrt{z}}, \qquad \frac{dz}{dx} = 2\,e^2 \sin x \cos x,$$

$$\therefore \quad \frac{dy}{dx} = \frac{e^2 \sin 2x}{2\sqrt{(1 + e^2 \sin^2 x)}}.$$

(44.) $$y = \log \tan x.$$

Here we have

$$y = \log z, \text{ where } z = \tan x,$$

$$\therefore \quad \frac{dy}{dz} = \frac{1}{z} = \frac{1}{\tan x},$$

and

$$\frac{dz}{dx} = \sec^2 x,$$

$$\therefore \quad \frac{dy}{dx} = \frac{\sec^2 x}{\tan x} = \frac{2}{\sin 2x}.$$

(45.) $$y = \frac{\sqrt{x^2+a^2}\ \sqrt[3]{x^2+b^2}}{\sqrt[5]{x^2+c^2}\ \sqrt[7]{x^2+d^2}},$$

$$\therefore \quad \log y = \tfrac{1}{2}\log(x^2+a^2) + \tfrac{1}{3}\log(x^2+b^2) - \tfrac{1}{5}\log(x^2+c^2) - \tfrac{1}{7}\log(x^2+d^2)$$

$$\therefore \quad \frac{1}{y}\frac{dy}{dx} = \frac{x}{x^2+a^2} + \frac{2x}{3(x^2+b^2)} - \frac{2x}{5(x^2+c^2)} - \frac{2x}{7(x^2+d^2)}$$

$$\therefore \frac{dy}{dx} = y\left\{\frac{x}{x^2+a^2} + \frac{2x}{3(x^2+b^2)} - \frac{2x}{5(x^2+c^2)} - \frac{2x}{7(x^2+d^2)}\right\}.$$

(46.) $$y = \frac{x \log (1 + x)}{1 + \sin x}.$$

By equation (F),

$$\frac{d\,y}{d\,x} = \frac{(1 + \sin x) \left\{ \log (1 + x) + \frac{x}{1 + x} \right\} - x \cos x \log (1 + x}{(1 + \sin x)^2}$$

(47.) $$y = \tan^{-1} \frac{\sqrt{x} + \sqrt{a} + \sqrt{b} - \sqrt{a\,b\,x}}{1 - \sqrt{a\,x} - \sqrt{b\,x} - \sqrt{a\,b}}.$$

Here

$$y = \tan^{-1} \sqrt{x} + \tan^{-1} \sqrt{a} + \tan^{-1} \sqrt{b},$$

$$\therefore \quad \frac{d\,y}{d\,x} = \frac{1}{2\sqrt{x}} \times \frac{1}{1 + x}.$$

(47*a*.) $$y = e^{k\,x},$$

$$\log y = k\,x,$$

$$\therefore \quad \frac{1}{y} \frac{d\,y}{d\,x} = k,$$

$$\therefore \quad \frac{d\,y}{d\,x} = k\,y = k\,e^{k\,x}.$$

(48.) $$y = e^{-k\,x},$$

$$\log y = -k\,x,$$

$$\therefore \quad \frac{1}{y} \frac{d\,y}{d\,x} = -k,$$

$$\therefore \quad \frac{d\,y}{d\,x} = -k\,y = -k\,e^{-k\,x}.$$

(49.) $$y = e^{k\,x} \sin a\,x.$$

By formula (C) we have

$$\frac{d\,y}{d\,x} = e^{k\,x}\, a \cos a\,x + e^{k\,x}\, k \sin a\,x,$$

$$= \sqrt{(a^2 + k^2)}\, e^{k\,x} \left\{ \frac{a \cos a\,x}{\sqrt{(a^2 + k^2)}} + \frac{k \sin a\,x}{\sqrt{(a^2 + k^2)}} \right\}.$$

Let θ denote the angle wh

$$\frac{a}{\sqrt{(a^2+k}}$$

then

$$\cos\theta = \frac{k}{\sqrt{(a^2+k^2)}}.$$

$$\therefore \quad \frac{dy}{dx} = e^{kx}\sqrt{(a^2+k^2)}\,\sin\,(a\,x$$

(50.) $$y = \sin a^{\frac{1}{x}},$$

therefore

$$y = \sin z, \text{ where } z = a^{\frac{1}{x}}.$$

Now

$$\frac{dy}{dz} = \cos z \text{ and } \frac{dz}{dx} = -\frac{a^{\frac{1}{x}}\log a}{x^2},$$

therefore, by formula (G),

$$\frac{dy}{dx} = -\frac{a^{\frac{1}{x}}\log a}{x^2}\cos a^{\frac{1}{x}}.$$

We add a list of differential coefficients for the sake of reference.

$y = x^n.$ $\quad \frac{dy}{dx} = n\,x^{n-1}.$

$y = \log_a x.$ $\quad \frac{dy}{dx} = \frac{\log_a e}{x}.$

$y = a^x.$ $\quad \frac{dy}{dx} = a^x \log a.$

$y = \sin a\,x.$ $\quad \frac{dy}{dx} = a\cos a\,x.$

$y = \cos a\,x.$ $\quad \frac{dy}{dx} = -a\sin a\,x.$

$y = \tan a\,x.$ $\quad \frac{dy}{dx} = a\sec^2 a\,x.$

$$\frac{}{x} = -a \operatorname{cosec}^2 a x.$$

$$\frac{d y}{d x} = \frac{1}{\sqrt{1 - x^2}}.$$

$$\frac{d y}{d x} = -\frac{1}{\sqrt{1 - x^2}}.$$

$$\frac{d y}{d x} = \frac{1}{1 + x^2}.$$

$^{-1} x.$ $$\frac{d y}{d x} = -\frac{1}{1 + x^2}.$$

$\sec^{-1} x.$ $$\frac{d y}{d x} = \frac{1}{x \sqrt{(x^2 - 1)}}.$$

$y = \operatorname{cosec}^{-1} x.$ $$\frac{d y}{d x} = -\frac{1}{x \sqrt{(x^2 - 1)}}.$$

$y = \operatorname{versin}^{-1} x.$ $$\frac{d y}{d x} = \frac{1}{\sqrt{2 x - x^2}}.$$

$y = \log (x + \sqrt{a^2 + x^2}).$ $$\frac{d y}{d x} = \frac{1}{\sqrt{(a^2 + x^2)}}.$$

$y = e^{k x}.$ $$\frac{d y}{d x} = k e^{k x}.$$

EXAMPLES.

1. $y = x^3. \quad \frac{d y}{d x} = 3 x^2.$

2. $y = 6 x^{\frac{1}{3}} \quad \frac{d y}{d x} = 2 x^{-\frac{2}{3}}.$

3. $y = a x^{-\frac{3}{4}}. \quad \frac{d y}{d x} = -\frac{3 a}{4} x^{-\frac{7}{4}}.$

4. $y = \sqrt[3]{(x^5)}. \quad \frac{d y}{d x} = \frac{5}{3} x^{\frac{2}{3}}.$

5. $y = x^2 - 3 x + 2. \quad \frac{d y}{d x} = 2 x - 3.$

6. $y = x^5 - x^4 - 3x - 1.$ $\quad \dfrac{dy}{dx} = 5x^4 - 4x^3 - 3.$

7. $y = x(x-1)(x-2).$ $\quad \dfrac{dy}{dx} = 3x^2 - 6x + 2.$

8. $y = \dfrac{x+a}{x+b}.$ $\quad \dfrac{dy}{dx} = \dfrac{b-a}{(x+b)^2}.$

9. $y = (x^3 - a^3)(b + x).$ $\quad \dfrac{dy}{dx} = 4x^3 + 3bx^2 - a^3.$

10. $y = (x+a)^n (x+b)^m.$ $\quad \dfrac{dy}{dx} = (x+a)^{n-1}(x+b)^{m-1}$

$\times \{(m+n)x + bn + am\}.$

11. $y = \sqrt{\dfrac{1+x^2}{1-x^2}}.$ $\quad \dfrac{dy}{dx} = \dfrac{2x}{(1-x^2)^{\frac{3}{2}} \sqrt{(1+x^2)}}.$

12. $y = \sin^2 x.$ $\quad \dfrac{dy}{dx} = \sin 2x.$ $\leftarrow 2 \sin x \cos x$

13. $y = \sin x^2.$ $\quad \dfrac{dy}{dx} = 2x \cos x^2.$

14. $y = \tan nx.$ $\quad \dfrac{dy}{dx} = n \sec^2 nx.$

15. $y = \tan x^n.$ $\quad \dfrac{dy}{dx} = n x^{n-1} \sec^2 x^n.$

16. $y = \sec^{-1} x^3.$ $\quad \dfrac{dy}{dx} = \dfrac{3}{x\sqrt{(x^6 - 1)}}.$

17. $y = \cos^{-1}(3x - 1).$ $\quad \dfrac{dy}{dx} = -\dfrac{\sqrt{3}}{\sqrt{x(2-3x)}}.$

18. $y = a\sqrt[3]{x}.$ $\quad \dfrac{dy}{dx} = \dfrac{a}{3\sqrt[3]{x^2}}.$

19. $y = x^4 - 3x^2 + 7.$ $\quad \dfrac{dy}{dx} = 4x^3 - 6x.$

20. $y = \frac{1-x}{1+x^2}$. $\frac{dy}{dx} = \frac{x^2 - 2x - 1}{(1+x^2)^2}$.

21. $y = \overline{(a^2 - x^2)}$. $\frac{dy}{dx} = -\frac{x}{\sqrt{a^2 - x^2}}$.

22. $y = (x+2)(x^2-4)$. $\frac{dy}{dx} = 3x^2 + 4x - 4$.

23. $y = \frac{e^x + e^{-x}}{e^x - e^{-x}}$. $\frac{dy}{dx} = -\frac{4}{(e^x - e^{-x})^2}$.

24. $y = \log(e^x - e^{-x})$. $\frac{dy}{dx} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$.

25. $y = \frac{c}{x^3}$. $\frac{dy}{dx} = -\frac{3c}{x^4}$.

26. $y = (1 + kx^3)^{\frac{1}{n}}$. $\frac{dy}{dx} = \frac{3kx^2}{n}(1+kx^3)^{\frac{1-n}{n}}$

27. $y = \log ax^2$. $\frac{dy}{dx} = \frac{2}{x}$.

28. $y = \log \sin x$. $\frac{dy}{dx} = \cot x$.

29. $y = a^x(1 + \log x)$. $\frac{dy}{dx} = a^x \Big\{(1 + \log x)\log a +$

30. $y = x^x e^x$. $\frac{dy}{dx} = x^x e^x (2 + \log x)$.

31. $y = \log(\sin^{-1} x)$. $\frac{dy}{dx} = \frac{1}{\sin^{-1} x \sqrt{(1-x^2)}}$.

32. $y = \frac{1}{a}\tan^{-1}\frac{x}{a}$. $\frac{dy}{dx} = \frac{1}{x^2 + a^2}$.

33. $y = \frac{1}{a}\log\frac{x}{a + \sqrt{a^2 \pm x^2}}$. $\frac{dy}{dx} = \frac{1}{x\sqrt{a^2 \pm x^2}}$.

34. $y = \sec a^x$. $\frac{dy}{dx} = a^x \log a \sec a^x \tan a^x$.

35. $y = \left(\frac{1}{x}\right)^x$. $\frac{dy}{dx} = \left(\frac{1}{x}\right)^x \left(\log\frac{1}{x} - 1\right)$.

36. $y = \sin x \sin^{-1} x$. $\frac{dy}{dx} = \frac{\sin x}{\sqrt{(1-x^2)}} + \sin^{-1} x \cos x$.

. $y = (p + qx)\, e^{ax}$. $\frac{dy}{dx} = e^{ax}\{q + a(p + qx)\}$.

8. $y = \sin^{-1}\sqrt{1 - x^2}$. $\frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}}$.

9. $y = \tan^{-1}\frac{3x}{1 - x^2}$. $\frac{dy}{dx} = \frac{3(1 + x^2)}{1 + 7x^2 + x^4}$.

1 $y = \log\sqrt{\left(\frac{1+x}{1-x}\right)}$. $\frac{dy}{dx} = \frac{1}{1 - x^2}$.

1 $y = \sin^{-1}\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}$. $\frac{dy}{dx} = \frac{\sqrt{1 - \sqrt{1 - x^2}}}{x\sqrt{2}\,(1 - x^2)^{\frac{3}{4}}}$.

1 $y = \frac{1}{x\log x}$. $\frac{dy}{dx} = -\frac{1 + \log x}{(x \log x)^2}$.

14. $y = \cos^{-1}\left(\frac{a + x}{2a + x}\right)$. $\frac{dy}{dx} = -\frac{a}{\sqrt{(3a^2 + 2ax)(2a + x)}}$.

15 $y = \frac{\sqrt{ax(bx + c)}}{\sqrt{x + a}}$. $\frac{dy}{dx} = \sqrt{\frac{ax(bx + c)}{(x + a)}}$

16 $\times \frac{1}{2}\left\{\frac{1}{x} + \frac{b}{bx + c} - \frac{1}{x + a}\right\}$.

17. $y = \{x + \sqrt{(x^2 - 1)}\}^n$. $\frac{dy}{dx} = n(x + \sqrt{x^2 - 1})^{n-1}$

18. $\times\left\{1 + \frac{x}{\sqrt{x^2 - 1}}\right\}$.

19. $y = \tan^{-1}\left(\frac{x + a^2}{a - ax}\right) + \log(x^2 + a^2)$. $\frac{dy}{dx} = \frac{a + 2x}{x^2 + a^2}$.

47. $y = (a^{\frac{1}{3}} + x^{\frac{1}{3}})^n (b^3 + x^3)^m.$ $\dfrac{d\,y}{d\,x} = (a^{\frac{1}{3}} + x^{\frac{1}{3}})^{n-1} (b^3 + x^3)^{m-1}$

$$\times \left\{3\,m\,x^2 (a^{\frac{1}{3}} + x^{\frac{1}{3}}) + \frac{n\,x^{-\frac{2}{3}}}{3} (b^3 + x^3)\right\}.$$

48. $y = e^{\frac{1}{x}}.$ $\dfrac{d\,y}{d\,x} = -\dfrac{e^{\frac{1}{x}}}{x^2}.$

49. $y = x^{\frac{x}{a}}.$ $\dfrac{d\,y}{d\,x} = \dfrac{x^{\frac{x}{a}}}{a}(1 + \log x).$

50. $y = \cos(\log \tan^{-1} x).$ $\dfrac{d\,y}{d\,x} = -\dfrac{\sin(\log \tan^{-1} x)}{\tan^{-1} x\,(1 + x^2)}.$

51. $y = \dfrac{x}{\sqrt{(1+x^2)}}.$ $\dfrac{d\,y}{d\,x} = \dfrac{1}{(1 + x^2)^{\frac{3}{2}}}.$

52. $y = \log \tan \dfrac{x}{2} - \dfrac{\cos x}{\sin^2 x}.$ $\dfrac{d\,y}{d\,x} = \dfrac{2}{\sin^3 x}.$

53. $y = \tan \dfrac{x}{2} \sin 2\,(x - a).$

$$\frac{d\,y}{d\,x} = \frac{2 \sin x \cos 2\,(x - a) + \sin 2\,(x - a)}{1 + \cos x}.$$

54. $y = x\,e^{3\,x}.$ $\dfrac{d\,y}{d\,x} = e^{3\,x}(1 + 3\,x).$

55. $y = \dfrac{x^{\frac{1}{3}} \sin \frac{x}{3}}{1 + x}.$ $\dfrac{d\,y}{d\,x} = \dfrac{x^{\frac{1}{3}} \cos \frac{x}{3} + x^{-\frac{2}{3}} \sin \frac{x}{3}}{3\,(1 + x)} - \dfrac{x^{\frac{1}{3}} \sin \frac{x}{3}}{(1 + x)^2}.$

56. $y = \left(\dfrac{x^x}{a^a}\right)^{\frac{1}{x-a}}.$

$$\frac{d\,y}{d\,x} = \left(\frac{x^x}{a^a}\right)^{\frac{1}{x-a}} \left\{\frac{(x - a)(1 + \log x) - \log\left(\frac{x^x}{a^a}\right)}{(x - a)^2}\right\}.$$

57. $y = x^x (1 - x)^{1 - x}.$ $\dfrac{d\,y}{d\,x} = x^x (1 - x)^{1 - x} \log\left(\dfrac{x}{1 - x}\right).$

58. $y = \dfrac{x\sqrt{a^2 - x^2}}{2} + \dfrac{a^2}{2}\sin^{-1}\left(\dfrac{x}{a}\right)$. $\quad \dfrac{dy}{dx} = \sqrt{(a^2 - x^2)}$.

59. $y = (\sin x \sin 2x \sin 3x)^n$.

$$\frac{dy}{dx} = ny(\cot x + 2\cot 2x + 3\cot 3x).$$

60. $y = \tan^{-1}(a^x)$. $\quad \dfrac{dy}{dx} = \dfrac{a^x \log a}{1 + a^{2x}}$.

61. $y = \sqrt{(ax + x^2)} + a\log(\sqrt{x} + \sqrt{a + x})$.

$$\frac{dy}{dx} = \sqrt{\left(\frac{x + a}{x}\right)}.$$

62. $y = \log\sqrt{\left(\dfrac{a\sin x + b\cos x}{a\sin x - b\cos x}\right)}$.

$$\frac{dy}{dx} = -\frac{ab}{a^2\sin^2 x - b^2\cos^2 x}.$$

63. $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.

$$\frac{dy}{dx} = -\frac{ax + hy + g}{hx + by + f}.$$

64. $y = \cfrac{\theta^2}{1 + \cfrac{\theta^2}{1 + \&c.}}$ $\quad \dfrac{dy}{d\theta} = \dfrac{\theta}{\sqrt{\theta^2 + \frac{1}{4}}}$.

65. $y = e^{\frac{y-x}{x}}$. $\quad \dfrac{dy}{dx} = \dfrac{(1 + \log y)^2}{\log y}$.

66. Given $y = a^4 b^3 x^2$, find the rate of change of y (1) with regard to a, (2) with regard to b, (3) with regard to x.

$$\frac{dy}{da} = 4a^3b^3x^2, \quad \frac{dy}{db} = 3a^4b^2x^2, \quad \frac{dy}{dx} = 2a^4b^3x.$$

CHAPTER II.

SUCCESSIVE DIFFERENTIATION.

ART. 12.—Successive Derived Functions and Differentials.

IN the preceding chapter we have shown how to find the first derived function from the primitive function.

If $y = f(x)$, we showed that the first derived function is $\frac{dy}{dx}$ or $\frac{df(x)}{dx}$, and that, with one exception, the first derived function differs from the primitive function; therefore, if

$$y = f(x),$$

on differentiating we have

$$\frac{dy}{dx} = \frac{df(x)}{dx} = f^{\text{I}}(x),$$

where $f^{\text{I}}(x)$ denotes the first derived function.

If $f^{\text{I}}(x)$ involve the variable x, we may treat it as we did $f(x)$, thus,

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{df^{\text{I}}(x)}{dx} = f^{\text{II}}(x),$$

the second derived function.

Instead of $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ we write $\frac{d^2y}{dx^2}$; therefore the result of differentiating $y = f(x)$ twice is denoted by

$$\frac{d^2y}{dx^2} = f^{\text{II}}(x).$$

Treating the second derived function in like manner if it contain the variable, we have

$$\frac{d^3 y}{d x^3} = f^{\text{III}}(x),$$

the third derived function.

Similarly, after n operations, we get

$$\frac{d^n y}{d x^n} = f^n(x),$$

the nth derived function.

We may arrange these results as follows:

$y = f(x)$. Primitive function.

$\frac{d y}{d x} = f^{\text{I}}(x)$. First derived function.

$\frac{d^2 y}{d x^2} = f^{\text{II}}(x)$. Second derived function.

$\frac{d^3 y}{d x^3} = f^{\text{III}}(x)$. Third derived function.

.. ..

$\frac{d^n y}{d x^n} = f^n(x)$. nth derived function.

(51.)

$$\begin{aligned} y &= x^3 - 4x^2 + 5x - 7 &&= f(x). \\ \frac{d y}{d x} &= 3x^2 - 8x + 5 &&= f^{\text{I}}(x). \\ \frac{d^2 y}{d x^2} &= 6x - 8 &&= f^{\text{II}}(x). \\ \frac{d^3 y}{d x^3} &= 6 &&= f^{\text{III}}(x). \\ \frac{d^4 y}{d x^4} &= 0. &&= f^{\text{IV}}(x). \end{aligned}$$

In this example we see that the derived functions of a higher order than the third vanish.

In like manner we may show that the $(n + 1)$th derived function of $a x^n + b x^{n-1} +$ &c. is zero, where n is a positive integer. But if n be negative or fractional there will be no derived function which will vanish, since by Rule I. we can never reduce the indices of x to cipher.

If $y = f(x)$
$d y = f^{\mathrm{I}}(x)\, d x$. The first differential.
$d^2 y = f^{\mathrm{II}}(x)\, d x^2$. The second differential.
$d^3 y = f^{\mathrm{III}}(x)\, d x^3$. The third differential.
.. ..
$d_n y = f^n(x)\, d x^n$. The nth differential.

(52.) Let $y = e^{kx}$,

then

$$\frac{d y}{d x} = k e^{kx}$$

$$\frac{d^2 y}{d x^2} = k^2 e^{kx}$$

.. ..

$$\frac{d^n y}{d x^n} = k^n e^{kx}.$$

Similarly, if

$$y = e^{-kx}$$

$$\frac{d y}{d x} = -k e^{-kx}$$

..

$$\frac{d^n y}{d x^n} = (-1)\ k^n e^{-kx}.$$

(53.) $y = \sin k t$

$$\frac{d y}{d t} = k \cos k t = k \sin\left(k t + \frac{\pi}{2}\right)$$

$$\frac{d^2 y}{d t^2} = k^2 \cos\left(k t + \frac{\pi}{2}\right) = k^2 \sin\left(k t + \frac{2\pi}{2}\right)$$

.. .. ..

$$\frac{d^n y}{d t^n} = k^n \sin\left(k t + \frac{n\pi}{2}\right).$$

(54.) $y = \cos k\,x$,

$$\frac{d\,y}{d\,x} = -\,k \sin k\,x = k \cos\left(k\,x + \frac{\pi}{2}\right),$$

$$\frac{d^2\,y}{d\,x^2} = -\,k^2 \sin\left(k\,x + \frac{\pi}{2}\right) = k^2 \cos\left(k\,x + \frac{2\,\pi}{2}\right),$$

$$\cdot\cdot \qquad \cdot\cdot \qquad \cdot\cdot$$

$$\frac{d^m\,y}{d\,x^m} = k^m \cos\left(k\,x + \frac{m\,\pi}{2}\right).$$

(55.)

$$y = \log_a x,$$

$$\frac{d\,y}{d\,x} = \frac{1}{x} \log_a e,$$

$$\frac{d^2\,y}{d\,x^2} = -\,\frac{1}{x^2} \log_a e,$$

$$\cdot\cdot \qquad \cdot\cdot$$

$$\frac{d^n\,y}{d\,x^n} = \frac{(\underline{|n - 1})\ (-\,1)^{n-1}}{x^n} \log_a e.$$

(56.)

$$y = e^{k\,x} \cos r\,x,$$

$$\frac{d\,y}{d\,x} = k\,e^{k\,x} \cos r\,x - r\,e^{k\,x} \sin r\,x$$

$$= e^{k\,x}\,(k \cos rx - r \sin rx)$$

$$= \sqrt{(k^2 + r^2)}\,e^{k\,x}\left(\frac{k}{\sqrt{k^2 + r^2)}} \cos r\,x - \frac{r}{\sqrt{k^2 + r^2}} \sin r\,x\right).$$

If

$$\frac{k}{\sqrt{k^2 + r^2}} = \cos\theta,$$

then

$$\frac{r}{\sqrt{k^2 + r^2}} = \sin\theta.$$

Substituting these values in the above expression we have

$$\frac{d\,y}{d\,x} = \sqrt{k^2 + r^2)}\,e^{k\,x}\,(\cos\theta \cos r\,x - \sin\theta \sin r\,x),$$

$$= \sqrt{(k^2 + r^2)}\,e^{k\,x} \cos(r\,x + \theta).$$

If we differentiate again we have

$$\frac{d^2 y}{d x^2} = \{\sqrt{(k^2 + r^2)}\}^2 e^{kx} \cos(r x + 2\theta)$$

Similarly, after differentiating n times we get

$$\frac{d^n y}{d x^n} = (k^2 + r^2)^{\frac{n}{2}} e^{kx} \cos(r x + n\theta).$$

If

$$V = R C + L \frac{d C}{d t}, \quad . \quad . \quad . \quad . \quad . \quad (a)$$

find V, where

$$C = a \sin k t.$$

Here

$$\frac{d C}{d t} = a k \cos k t.$$

Substituting in (a) we have

$$V = R a \sin k t + L a k \cos k t$$

$$= \sqrt{(R a)^2 + (L a k)^2} \left\{ \frac{R a \sin k t}{\sqrt{(R a)^2 + (L a k)^2}} + \frac{L a k \cos k t}{\sqrt{(R a)^2 + (L a k)^2}} \right\}$$

$$= \sqrt{(R a)^2 + (L a k)^2} \sin(k t + \theta),$$

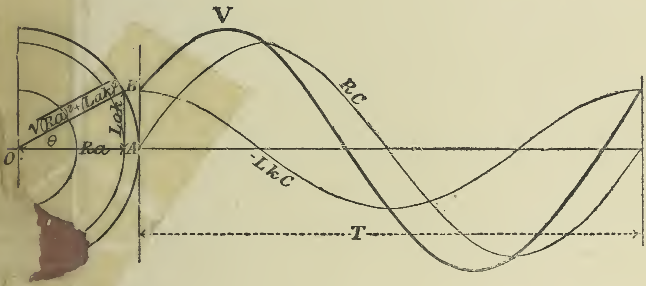
where

$$\theta = \tan^{-1} \frac{L a k}{R a}.$$

In a later chapter we shall show how to find C in terms of V, R, L and t.

Art. 13.—In the preceding example, V stands for the electromotive-force that must be impressed on an electric circuit in order to produce a current of C amperes, L stands for the coefficient of self-induction or back electromotive-force when the current is growing at the rate of one ampere in one second, and k is $2\pi \times$ *frequency* or $\frac{2\pi}{T}$ where T is the periodic time in seconds.

In order to represent this graphically, plot a sine curve (*see the author's "Elementary Practical Mathematics," page* 163) whose amplitude is $R\,a$, to represent the electromotive-force required to drive the current through the resistance if there were no inductance in the circuit. Then on the same time base plot a cosine curve whose amplitude is $L\,a\,k$ to represent the electromotive-force necessary to balance the reaction of self-induction. Also plot a curve whose amplitude is $\sqrt{(R\,a)^2 + (L\,a\,k)^2}$, as shown in the accompanying figure.



The V curve represents the electromotive-force that must be impressed on the circuit in order to produce the current C.

In the figure $O\,A = R\,a$, $A\,B = L\,a\,k$, $O\,B = \sqrt{(R\,a)^2 + (L\,a\,k)^2}$, and the angle $A\,O\,B$ is the angle of lag or amount by which the current lags behind the electromotive-force.

The expression $\sqrt{R^2 + (L\,k)^2}$ is called the *impedence.*

(57.) Example. Given $R = 2$, $L = \cdot002$, $C = 50$ virtual amperes, and $T = \frac{1}{50}$, find V, the virtual volts that must be mpressed on the circuit, k, angle of lag θ, and the mpedence.

Here the virtual volts

$$V = \sqrt{2^2 + (\cdot 628)^2} \times 50;$$

therefore

$$V = 104\cdot 8 \text{ nearly},$$

and

$$k = \frac{2\pi}{T} = 100 \times 3\cdot 1416 = 314\cdot 16.$$

Also

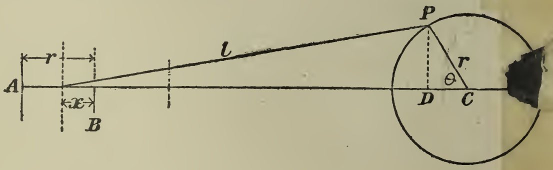
$$\tan\theta = \frac{L\,k}{R} = \frac{\cdot 628}{2} = \cdot 319;$$

therefore

$$\theta = 17^\circ\ 12'.$$

$$\text{Impedence} = \sqrt{2^2 + (\cdot 628)^2} = 2\cdot 096.$$

ART. 13A.—In the accompanying figure, A represents th position of the cross-head of a steam engine when the crank is at a dead point, B is the middle of the stroke, and x is the displacement of the cross-head from its mean positio when the crank angle is θ.



Now

$$A\,C = l + r = r - x + \sqrt{l^2 - r^2 \sin^2\theta} + r\cos\theta;$$

whence

$$x = r\cos\theta + \sqrt{l^2 - r^2\sin^2\theta} - l \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where l is the length of the connecting rod, r is the crank, and θ the crank angle, measured from the dead centre.

Suppose we require the piston velocity for any give value of θ, we have to find the value of $\frac{d\,x}{d\,t}$, which is th velocity, where t denotes the time reckoned from the instan the crank passes the dead centre.

From (1) we have

$$d\,x = -r \sin\theta\, d\,\theta - \frac{r^2 \sin\theta \cos\theta}{\sqrt{l^2 - r^2 \sin^2\theta}}\, d\,\theta,$$

that is the change in x corresponding to a very small change in θ; but velocity is rate of change of displacement, therefore

$$\text{velocity} = \frac{d\,x}{d\,t} = -r \sin\theta \left(\frac{d\,\theta}{d\,t}\right) - \frac{r^2 \sin\theta \cos\theta}{\sqrt{l^2 - r^2 \sin^2\theta}} \left(\frac{d\,\theta}{d\,t}\right) \quad . \quad . \quad (2)$$

The piston acceleration is the rate of change of velocity, consequently we have to differentiate (2) with regard to time t, therefore

$$\text{acceleration } \frac{d^2\,x}{d\,t^2} = -\,r \cos\theta \left(\frac{d\,\theta}{d\,t}\right)^2$$
$$-\frac{r^2\,\{l^2 \cos 2\,\theta + r^2 \sin^4\theta\}}{(l^2 - r^2 \sin^2\theta)^{\frac{3}{2}}} \left(\frac{d\,\theta}{d\,t}\right)^2.$$

This is assuming that the angular velocity $\frac{d\,\theta}{d\,t}$ of the crank is constant.

EXAMPLES.

1. If $y = x^4$, show that $\dfrac{d^3\,y}{d\,x^3} = 24\,x$.
2. If $y = x^{\frac{1}{2}}$, ,, $\dfrac{d^2\,y}{d\,x^2} = -\,\frac{1}{4}\,x^{-\frac{3}{2}}$.
3. If $y = x^{-\frac{3}{2}}$, ,, $\dfrac{d^4\,y}{d\,x^4} = \dfrac{945}{16}\,x^{-\frac{11}{2}}$.
4. If $y = \sin x$, ,, $\dfrac{d^6\,y}{d\,x^6} = -\sin x$.
5. If $y = \cos x^2$, ,, $\dfrac{d^3\,y}{d\,x^3} = 8\,x^3 \sin x^2 - 12\,x \cos x^2$.
6. If $y = x \sin x$, ,, $\dfrac{d^5\,y}{d\,x^5} = 5 \sin x + x \cos x$.

7. If $y = x^n e^x$, show that

$$\frac{d^2 y}{d x^2} = e^x \{n (n - 1) x^{n-2} + 2 n x^{n-1} + x^n\}.$$

8. If $y = e^x \tan^{-1} x$, show that

$$\frac{d^2 y}{d x^2} = \frac{(1 - x)^2 e^x}{(1 + x^2)^2} + \frac{e^x}{1 + x^2} + y.$$

9. If $y = e^{x^2}$, show that

$$\frac{d^6 y}{d x^6} = e^{x^2} (64 x^6 + 480 x^4 + 720 x^2 + 120).$$

10. If $y = \log (\cos x)$, show that

$$\frac{d^3 y}{d x^3} = - 2 \sec^2 x \tan x.$$

11. If $y = e^x \sin x$, show that

$$\frac{d^4 y}{d x^4} + 4 y = 0.$$

12. If $y = e^{ax} \sin x$, show that

$$\frac{d^n y}{d x^n} = \frac{e^{ax} \sin (x + n \theta)}{\sin^n \theta},$$

where $\tan \theta = \dfrac{1}{a}$.

13. If $y = (x^2 + n^2) \tan^{-1} \left(\dfrac{x}{n}\right)$, show that

$$\frac{d^3 y}{d x^3} = \frac{4 n^3}{(n^2 + x^2)^2}.$$

14. If $y = C \sin k x + C_1 \cos k x$, show that

$$\frac{d^4 y}{d x^4} = k^4 y.$$

15 If $y = e^{k \sin^{-1} x}$, show that

$$(1 - x^2) \frac{d^2 y}{d x^2} - x \frac{d y}{d x} = k^2 y.$$

16. If $y = k \cos (\log x) + l \sin (\log x)$, show that

$$x^2 \frac{d^2 y}{d x^2} + x \frac{d y}{d x} + y = 0.$$

17. $y = \dfrac{n^2}{n^2 + x^2}$.

$$\frac{d^5 y}{d x^5} = \frac{- 48 n^2 x (9 n^4 - 26 n^2 x^2 + 5 x^4)}{(n^2 + x^2)^6}.$$

18. Find the velocity and acceleration of the piston of a steam engine from the following data :—

Crank, 12 inches ; connecting rod, 5 feet ; crank angle, 30° ; speed, 100 revolutions per minute.

The angular velocity $\dfrac{d \theta}{d t} = \dfrac{10 \pi}{3}$ radians per second.

Velocity = 368·6 feet per minute.
Acceleration = 11·94 feet per second per second.

CHAPTER III.

EXPANSION OF FUNCTIONS.

ART. 14.—Taylor's and Maclaurin's Theorems.

If $f\,(x + y)$ be a continuous function of $x + y$ whi admits of being expressed in the form of a series of ascendir integral powers of x or y; then

$$f(x + y) = A + B\,y + C\,y^2 + D\,y^3 + \&c.$$

or

$$f(x + y) = A_{,} + B_{,}\,x + C_{,}\,x^2 + D_{,}\,x^3 + \&c.,$$

where A, B, C, D, &c., are functions of x and independent of y, and $A_{,}$, $B_{,}$, $C_{,}$, $D_{,}$, &c., are functions of y and independent of x. For example:

$$(x + y)^n = x^n + n\,x^{n-1}\,y + \frac{n\,(n - 1)}{\underline{|2}}\,x^{n-2}\,y^2 + \&c.$$

$$= A + B\,y + C\,y^2 + \&c.$$

where

$$A = x^n, \quad B = n\,x^{n-1}, \quad C = \frac{n\,(n-1)}{\underline{|2}}\,x^{n-2}, \&c.$$

We shall now show how the values of A, B, C, D, &c., o $A_{,}$, $B_{,}$, $C_{,}$, $D_{,}$, &c., may be obtained in terms of x or y resp tively.

If $f\,(x + y)$ be a function which admits of being expan in powers of x or y; on differentiating it with respect tc treating y as constant, we get a result which is the sam if we were to differentiate it with regard to y treating constant.

For the proof of this statement,

Let

$$u = f(x + y) = f(z),$$

where

$$z = x + y.$$

Then

$$\frac{du}{dz} = f'(z),$$

but

$$\frac{du}{dx} = \frac{du}{dz}\,\frac{dz}{dx}$$

and since

$$z = x + y,$$

we have

$$\frac{dz}{dx} = 1 \text{ treating } y \text{ as a constant,}$$

therefore

$$\frac{du}{dx} = f'(z).$$

Similarly,

$$\frac{du}{dy} = \frac{du}{dz}\,\frac{dz}{dy} = f'(z).$$

Since

$$\frac{dz}{dy} = 1 \text{ treating } x \text{ as a constant,}$$

therefore

$$\frac{du}{dx} = \frac{du}{dy}, \text{ where } u = f(x + y).$$

For example: Let

$$u = (x + y)^4,$$

therefore

$$\frac{du}{dx} = 4(x + y)^3 \text{ treating } y \text{ as constant,}$$

and

$$\frac{du}{dy} = 4(x + y)^3 \text{ treating } x \text{ as constant;}$$

therefore

$$\frac{d\,u}{d\,x} = \frac{d\,u}{d\,y}.$$

Let

$$u = f(x + y) = \mathrm{A} + \mathrm{B}\,y + \mathrm{C}\,y^2 + \mathrm{D}\,y^3 + \&\text{c.} \quad . \quad (1)$$

On differentiating this with regard to x, we get

$$\frac{d\,u}{d\,x} = \frac{d\,\mathrm{A}}{d\,x} + \frac{d\,\mathrm{B}}{d\,x}\,y + \frac{d\,\mathrm{C}}{d\,x}\,y^2 + \frac{d\,\mathrm{D}}{d\,x}\,y^3 + \&\text{c.}\,; \quad . \quad (2)$$

also

$$\frac{d\,u}{d\,y} = \mathrm{B} + 2\,\mathrm{C}\,y + 3\,\mathrm{D}\,y^2 + \&\text{c.};$$

and since we have proved that

$$\frac{d\,u}{d\,x} = \frac{d\,u}{d\,y},$$

we are at liberty to assume that the coefficients of like powers of y are equal in the above equations, therefore

$$\frac{d\,\mathrm{A}}{d\,x} = \mathrm{B}, \quad \frac{d\,\mathrm{B}}{d\,x} = 2\,\mathrm{C}, \quad \frac{d\,\mathrm{C}}{d\,x} = 3\,\mathrm{D}, \ \&\text{c.}$$

Again, since (1) holds for all values of y, it must hold when $y = 0$, therefore

$$f(x) = \mathrm{A}.$$

Again,

$$\mathrm{B} = \frac{d\,\mathrm{A}}{d\,x} = f^{\mathrm{I}}(x),$$

and

$$\mathrm{C} = \frac{1}{2}\,\frac{d\,\mathrm{B}}{d\,x} = \frac{1}{\underline{|2}}\,f^{\mathrm{II}}(x).$$

Similarly,

$$\mathrm{D} = \frac{1}{\underline{|3}}\,f^{\mathrm{III}}(x), \ \&\text{c.}$$

On substituting these values of A, B, C, D, &c., in (1) we have

$$f(x+y) = f(x) + y f^{\mathrm{I}}(x) + \frac{y^2}{\lfloor 2} f^{\mathrm{II}}(x) + \frac{y^3}{\lfloor 3} f^{\mathrm{III}}(x) + \&c.$$

This result is called **Taylor's Theorem.**

Similarly, we may show that

$$f(x+y) = f(y) + x f^{\mathrm{I}}(y) + \frac{x^2}{\lfloor 2} f^{\mathrm{II}}(y) + \frac{x^3}{\lfloor 3} f^{\mathrm{III}}(y) + \&c.$$

ART. 15.—We shall now apply *Taylor's Theorem* in expanding a few functions of $x + y$.

(58.) **Expansion of** $(x+y)^6$.

$$(x+y)^6 = f(x) + y f^{\mathrm{I}}(x) + \frac{y^2}{\lfloor 2} f^{\mathrm{II}}(x) + \frac{y^3}{\lfloor 3} f^{\mathrm{III}}(x) + \&c.$$

Here

$$f(x) = x^6, \quad f^{\mathrm{I}}(x) = 6\,x^5, \quad f^{\mathrm{II}}(x) = 6 \times 5\,x^4, \&c.$$

On substituting these values for the several successive derived functions of x, and simplifying the coefficients, we have

$$(x+y)^6 = x^6 + 6\,x^5 y + 15\,x^4 y^2 + 20\,x^3 y^3 \\ + 15\,x^2 y^4 + 6\,x y^5 + y^6.$$

(59.) **To expand** $\log(x+y)$.

$$\mathrm{Log}\,(x+y) = f(x) + y f^{\mathrm{I}}(x) + \frac{y^2}{\lfloor 2} f^{\mathrm{II}}(x) + \frac{y^3}{\lfloor 3} f^{\mathrm{III}}(x) + \&c.$$

Here

$$f(x) = \log x, f^{\mathrm{I}}(x) = \frac{1}{x}, f^{\mathrm{II}}(x) = -\frac{1}{x^2}, f^{\mathrm{III}}(x) = \frac{2}{x^3}, \&c.$$

On substituting these values for the several derived functions, we get

$$\log(x+y) = \log x + \frac{y}{x} - \frac{y^2}{2\,x^2} + \frac{y^3}{3\,x^3} - \frac{y^4}{4\,x^4} + \&c.$$

(59*a*.) **To expand** sin $(x+y)$.

$$\text{Sin}\,(x+y)=f(x+y)=f(x)+yf^{\text{I}}(x)+\frac{y^2}{\underline{|2}}f^{\text{II}}(x)$$
$$+\frac{y^3}{\underline{|3}}f^{\text{III}}(x)+\&\text{c.}$$

Here

$f(x)=\sin x$, $f^{\text{I}}(x)=\cos x$, $f^{\text{II}}(x)=-\sin x$, &c.

On substituting these values for the several derived functions of x, we obtain

$$\text{Sin}\,(x+y)=\sin x+y\cos x-\frac{y^2}{\underline{|2}}\sin x-\frac{y^3}{\underline{|3}}\cos x$$
$$+\frac{y^4}{\underline{|4}}\sin x+\frac{y^5}{\underline{|5}}\cos x-\&\text{c.},$$

or thus

$$\text{Sin}\,(x+y)=\sin x\left(1-\frac{y^2}{\underline{|2}}+\frac{y^4}{\underline{|4}}-\frac{y^6}{\underline{|6}}+\&\text{c.}\right)$$
$$+\cos x\left(y-\frac{y^3}{\underline{|3}}+\frac{y^5}{\underline{|5}}-\frac{y^7}{\underline{|7}}+\&\text{c.}\right).$$

(60.) **To expand** cos $(x+y)$.

$$\text{Cos}\,(x+y)=f(x+y)=f(x)+yf^{\text{I}}(x)+\frac{y^2}{\underline{|2}}f^{\text{II}}(x)+\&\text{c.}$$

Here

$f(x)=\cos x$, $f^{\text{I}}(x)=-\sin x$, $f^{\text{II}}(x)=-\cos x$, &c.

On substituting, we obtain

$$\text{Cos}\,(x+y)=\cos x-y\sin x-\frac{y^2}{\underline{|2}}\cos x+\frac{y^3}{\underline{|3}}\sin x+\&\text{c.}$$
$$=\cos x\left(1-\frac{y^2}{\underline{|2}}+\frac{y^4}{\underline{|4}}-\frac{y^6}{\underline{|6}}+\&\text{c.}\right)$$
$$-\sin x\left(y-\frac{y^3}{\underline{|3}}+\frac{y^5}{\underline{|5}}-\frac{y^7}{\underline{|7}}+\&\text{c.}\right).$$

(61.) **To expand** a^{x+y}.

$$a^{x+y}=f(x+y)=f(x)+yf^{\text{I}}(x)+\frac{y^2}{\underline{|2}}f^{\text{II}}(x)+\&\text{c.}$$

Here

$$f(x) = a^x,\ f^{\text{I}}(x) = a^x \log a,\ f^{\text{II}}(x) = a^x (\log a)^2, \text{ \&c.}$$

Therefore

$$a^{x+y} = a^x + y\, a^x \log a + \frac{(y \log a)^2}{\underline{|2}}\, a^x + \text{\&c.}$$

$$= a^x\left(1 + y \log a + \frac{(y \log a)^2}{\underline{|2}} + \frac{(y \log a)^3}{\underline{|3}} + \text{\&c.}\right).$$

Art. 16.—Maclaurin's Theorem.

In the expansion of $f(x + y)$, by *Taylor's Theorem*, if we put $x = 0$, we have then

$$f(y) = f(0 + y) = f(0) + y\, f^{\text{I}}(0) + \frac{y^2}{\underline{|2}} f^{\text{II}}(0)$$

$$+ \frac{y}{\underline{|3}} f^{\text{III}}(0) + \text{\&c.}$$

Similarly, we may show that

$$f(x) = f(0) + x\, f^{\text{I}}(0) + \frac{x^2}{\underline{|2}} f^{\text{II}}(0) + \frac{x^3}{\underline{|3}} f^{\text{III}}(0) + \text{\&c.}$$

This result is called **Maclaurin's Theorem.**

The beginner has usually some difficulty in interpreting the meaning of $f(0)$, $f'(0)$, $f''(0)$, &c.

Now

$f(0)$ is what $f(x)$ becomes when $x = 0$.
$f^{\text{I}}(0)$,, $f^{\text{I}}(x)$,, ,,
$f^{\text{II}}(0)$,, $f^{\text{II}}(x)$,, ,,

(62.) Let us take $\sin x$.

Here

$f(x) = \sin x. \quad \therefore f(0) = \sin 0 = 0.$

and

$f^{\text{I}}(x) = \cos x. \quad \therefore f^{\text{I}}(0) = \cos 0 = 1.$
$f^{\text{II}}(x) = -\sin x. \quad \therefore f^{\text{II}}(0) = -\sin 0 = 0.$
$f^{\text{III}}(x) = -\cos x. \quad \therefore f^{\text{III}}(0) = -\cos 0 = -1.$

$$\therefore \quad \text{Sin}\ x = f(x) = f(0) + x\,f^{\text{I}}(0) + \frac{x^2}{\underline{|2}}\,f^{\text{II}}(0)$$
$$+ \frac{x^3}{\underline{|3}}\,f^{\text{III}}(0) + \&c.$$
$$= 0 + x \times 1 + \frac{x^2}{\underline{|2}} \times 0 + \frac{x^3}{\underline{|3}} \times (-1) + \frac{x^4}{\underline{|4}} \times 0 + \&c.$$
$$\therefore \quad \sin x = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} + \&c.$$

(63.) $\text{Cos}\ x = f(x) = f(0) + x\,f^{\text{I}}(0) + \frac{x^2}{\underline{|2}}\,f^{\text{II}}(0)$
$$+ \frac{x^3}{\underline{|3}}\,f^{\text{III}}(0) + \&c.$$

Here

$f(x) = \cos x. \qquad \therefore f(0) = \cos 0 = 1$

and

$f^{\text{I}}(x) = -\sin x. \qquad \therefore f^{\text{I}}(0) = -\sin 0 = 0.$
$f^{\text{II}}(x) = -\cos x. \qquad \therefore f^{\text{II}}(0) = -\cos 0 = -1.$

$$\therefore \quad \cos x = 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \frac{x^6}{\underline{|6}} + \&c.$$

(64.) Exponential Theorem.

To expand a^x.

Here

$$a^x = f(x) = f(0) + x\,f^{\text{I}}(0) + \frac{x^2}{\underline{|2}}\,f^{\text{II}}(0) + \frac{x^3}{\underline{|3}}\,f^{\text{III}}(0) + \&c.$$

Now

$$f(0) = a^0 = 1, \quad f^{\text{I}}(0) = a^0 \log a = \log a,$$
$$f^{\text{II}}(0) = a^0 (\log a)^2 = (\log a)^2, \ \&c.$$

$$\therefore \ a^x = 1 + x \log a + \frac{x^2}{\underline{|2}} (\log a)^2 + \frac{x^3}{\underline{|3}} (\log a)^3 + \&c.$$

If we substitute e, the base of the Napierian logarithms, for a, we get

$$e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \&c., \text{ since } \log_e e = 1.$$

(65.) **To expand** $\tan x$.

$$f(x) = \tan x = f(0) + x f^{\mathrm{I}}(0) + \frac{x^2}{\lfloor 2} f^{\mathrm{II}}(0) + \frac{x^3}{\lfloor 3} f^{\mathrm{III}}(0)$$
$$+ \frac{x^4}{\lfloor 4} f^{\mathrm{IV}}(0) + \&c.$$

$f(x) = \tan x. \quad \therefore f(0) = \tan(0) = 0.$
$f^{\mathrm{I}}(x) = 1 + \tan^2 x. \quad \therefore f^{\mathrm{I}}(0) = 1 + \tan^2(0) = 1.$
$f^{\mathrm{II}}(x) = 2 \tan x + 2 \tan^3 x. \quad \therefore f^{\mathrm{II}}(0) = 2 \tan(0) + 2 \tan^3(0) = 0.$

$f^{\mathrm{III}}(x) = 2 + 8 \tan^2 x + 6 \tan^4 x. \quad \therefore f^{\mathrm{III}}(0) = 2,$
$f^{\mathrm{IV}}(x) = 16 \tan x + 40 \tan^3 x + 24 \tan^5 x. \quad \therefore f^{\mathrm{IV}}(0) = 0.$
$f^{\mathrm{V}}(x) = 16 +$ terms involving $\tan x$. $\therefore f^{\mathrm{V}}(0) = 16$, &c.

Therefore

$$\tan x = 0 + x + \frac{x^2}{\lfloor 2} \times 0 + \frac{x^3}{\lfloor 3} \times 2 + \frac{x^4}{\lfloor 4} \times 0 + \frac{x^5}{\lfloor 5} \times 16 + \&c;$$

that is

$$\tan x = x + \frac{x^3}{3} + \frac{2\,x^5}{15} + \&c.$$

(66.) **Expansion of** $\log(1 + x)$.

$$f(x) = \log(1 + x) = f(0) + x f^{\mathrm{I}}(0) + \frac{x^2}{\lfloor 2} f^{\mathrm{II}}(0)$$
$$+ \frac{x^3}{\lfloor 3} f^{\mathrm{III}}(0) + \&c.$$

Here

$$f(0) = \log 1 = 0.$$
$$f^{\mathrm{I}}(0) = \frac{1}{1 + 0} = 1.$$
$$f^{\mathrm{II}}(0) = -\frac{1}{(1 + 0)^2} = -1.$$
$$f^{\mathrm{III}}(0) = \frac{2}{(1 + 0)^3} = 2.$$

$$\therefore \log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \&c.$$

In this result let x be changed to $-x$.

$$\therefore \quad \log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \&c.$$

Art. 17.—Euler's Formulæ for Sine and Cosine.

If we expand $e^{x\sqrt{-1}}$ by Maclaurin's Theorem we obtain

$$e^{x\sqrt{-1}} = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \&c. \; . \; . \; .$$

$$+ \sqrt{-1}\left\{x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \&c.\right\}.$$

$$= \cos x + \sqrt{-1}\sin x, \text{ by (62), (63).}$$

Change x to $-x$, and we have

$$e^{-x\sqrt{-1}} = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \&c.$$

$$- \sqrt{-1}\left\{x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \&c.\right\}$$

$$= \cos x - \sqrt{-1}\sin x.$$

On adding and dividing by 2, we get

$$\cos x = \frac{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}{2}.$$

Subtracting, we get

$$\sin x = \frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{2\sqrt{-1}}.$$

Art. 18.*—Hyperbolic Functions.

Let P B Q be a hyperbolic curve whose equation is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

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[*] The student may omit this article till he has mastered the elements of the integral calculus.

It is easy to prove that the sectorial area O B P

$$= \frac{a\,b}{2} \log \left(\frac{x}{a} + \frac{y}{b}\right).$$

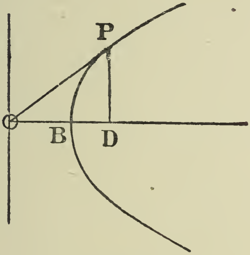

FIG. 5.

$$\therefore \quad \log \left(\frac{x}{a} + \frac{y}{b}\right) = \frac{2\,\text{O B P}}{a\,b} = u, \text{ say.}$$

$$\therefore \quad \frac{x}{a} + \frac{y}{b} = e^{u} \quad . \quad . \quad . \quad . \quad . \quad (a)$$

where *e* is the base of the Napierian logarithms.

By the equation of the curve we have

$$\left(\frac{x}{a} + \frac{y}{b}\right)\left(\frac{x}{a} - \frac{y}{b}\right) = 1. \qquad \therefore \quad e^{u}\left(\frac{x}{a} - \frac{y}{b}\right) = 1.$$

$$\therefore \quad \frac{x}{a} - \frac{y}{b} = e^{-u}. \quad . \quad . \quad . \quad . \quad (b)$$

Adding (*a*) and (*b*) we obtain

$$\frac{x}{a} = \tfrac{1}{2}\left(e^{u} + e^{-u}\right),$$

and subtracting, we have

$$\frac{y}{b} = \tfrac{1}{2}\left(e^{u} - e^{-u}\right).$$

The former is called the hyperbolic cosine of u, and th latter the hyperbolic sine of u, and denoted by $\cos h\, u$ an sin $h\, u$ respectively.

(67.) $$\text{Cos}\, h\, u = \tfrac{1}{2}(e^u + e^{-u}).$$

$$\therefore \frac{d(\cos h\, u)}{d\, u} = \tfrac{1}{2}(e^u - e^{-u}) = \sin h\, u = \sqrt{\cos h^2 u - 1}$$

Similarly,

$$\frac{d(\sin h\, u)}{d\, u} = \tfrac{1}{2}(e^u + e^{-u}) = \cos h\, u = \sqrt{\sin h^2 u + 1}.$$

Also $\cos h^2 u - \sin h^2 u = 1$.

Again, let $\cos h\, u = x$.

$$\therefore \frac{d\,x}{d\,u} = \sqrt{(x^2 - 1)}. \qquad \therefore \frac{d\,u}{d\,x} = \frac{1}{\sqrt{x^2 - 1}}.$$

$$\therefore u = \int_1^x \frac{d\,x}{\sqrt{x^2 - 1}} = \log(x + \sqrt{x^2 - 1}).$$

$$\therefore \cos h^{-1} x = \log(x + \sqrt{x^2 - 1}).$$

Similarly, we can show that

$$\sin h^{-1} x = \log(x + \sqrt{x^2 + 1}).$$

(68.) **To expand** $\tan^{-1} x$.

Assume $\tan^{-1} x = a + b\,x + c\,x^2 + d\,x^3 + \&c.$, and di rentiate both sides of this equation with regard to x, the fore

$$\frac{1}{1 + x^2} = b + 2\,c\,x + 3\,d\,x^2 + \&c. \quad . \quad . \quad (\alpha)$$

Now

$$\frac{1}{1 + x^2} = 1 - x^2 + x^4 - x^5 + \&c.; \quad . \quad . \quad (\beta)$$

and since (α) and (β) are identical, we may assume that the coefficients of like powers of x are equal in both expressions therefore $b = 1$, $c = 0$, $d = -\tfrac{1}{3}$, and $a = 0$, since $\tan^{-1}(0) = 0$.

$$\therefore \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \&c.$$

(69.) **To expand** $\sin^{-1} x$.

Assume

$$\sin^{-1} x = a + b\,x + c\,x^2 + d\,x^3 + e\,x^4 + \&c. \quad (\alpha)$$

Differentiate both sides; thus

$$\frac{1}{(1 - x^2)^{\frac{1}{2}}} = b + 2\,c\,x + 3\,d\,x^2 + 4\,e\,x^3 + \&c. \quad (\beta)$$

But

$$\frac{1}{(1 - x^2)^{\frac{1}{2}}} = 1 + \tfrac{1}{2}\,x^2 + \frac{1 \times 3}{2 \times 4}\,x^4 + \frac{1 \times 3 \times 5}{2 \times 4 \times 6}\,x^6 + \frac{1 \times 3 \times 5 \times 7}{2 \times 4 \times 6 \times 8}\,x^8 + \&c. \quad . \quad . \quad . \quad (\gamma)$$

Equating the coefficients of like powers of x in (β) and (γ) we get $b = 1$, $c = 0$, $d = \dfrac{1}{2 \times 3}$, $e = 0$, &c., and $a = 0$, $\sin^{-1}(0) = 0$.

On substituting these values for a, b, c, d, &c., in (α) we have

$$\sin^{-1} x = x + \frac{1}{2 \times 3}\,x^3 + \frac{1 \times 3}{2 \times 4 \times 5}\,x^5 + \frac{1 \times 3 \times 5}{2 \times 4 \times 6 \times 7}\,x^7 + \&c.$$

In the above trigonometrical expansions the angle x is timated in circular measure, and in order that the expanons may form rapidly convergent series, the value of x must less than unity. We add a few numerical examples:—

(70.) **To find the sine and cosine of 10°.**

We must express 10° in radians, that is, $\dfrac{\pi}{18}$ radians; herefore by (62),

$$\sin 10^\circ = \frac{\pi}{18} - \frac{1}{\underline{\lvert 3}}\left(\frac{\pi}{18}\right)^3 + \frac{1}{\underline{\lvert 5}}\left(\frac{\pi}{18}\right)^5 - \&c.$$

$$\therefore \quad \sin 10^\circ = \cdot 1736482 \ldots$$

$$\cos 10° = 1 - \frac{1}{\underline{|2}}\left(\frac{\pi}{18}\right)^2 + \frac{1}{\underline{|4}}\left(\frac{\pi}{18}\right)^4 - \&c.$$

$$\therefore \quad \cos 10° = \cdot 9848078 \ldots$$

(71.) To find the tangent of 24°.

By (65)

$$\tan 24° = \frac{2\pi}{15} + \frac{1}{3}\left(\frac{2\pi}{15}\right)^3 + \frac{2}{15}\left(\frac{2\pi}{15}\right)^5 + \&c.$$

$$\therefore \quad \tan 24° = \cdot 4452287 \ldots .$$

By means of examples (70) we can find the sine or cosine of any angle less than one radian; but we can also make use of it to find the sine or cosine of an angle greater than one radian.

Suppose we want the sine of 78°. Now

$$\sin 78° = \cos 12° = 1 - \frac{1}{\underline{|2}}\left(\frac{\pi}{15}\right)^2 + \frac{1}{\underline{|4}}\left(\frac{\pi}{15}\right)^4 - \frac{1}{\underline{|6}}\left(\frac{\pi}{15}\right)^6 + \&c.$$

$$= \cdot 9781476.$$

EXAMPLES.

1. Expand $(1 + x)\, e^x$ by Maclaurin's Theorem to six terms.

$$\textit{Ans.} \quad 1 + 2x + \frac{3x^2}{2} + \frac{2x^3}{3} + \frac{5x^4}{24} + \frac{x^5}{20} + \&c.$$

2. Expand $e^x \sin x$ in powers of x.

$$\textit{Ans.} \quad x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} - \frac{x^6}{90} - \frac{x^7}{630} + \&c.$$

3. Expand to five terms $e^x \sec x$.

$$\textit{Ans.} \quad 1 + x + x^2 + \frac{2x^3}{3} + \frac{x^4}{2} + \&c.$$

4. Expand $\log(1 + e^{ax})$ to four terms.

$$\textit{Ans.} \quad \text{Log } 2 + \frac{ax}{2} + \frac{a^2x^2}{8} - \frac{a^4x^4}{192} \ldots \&c.$$

5. Expand $e^{\sin^{-1}x}$ to five terms in powers of x.

$$\textit{Ans.}\quad 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{5\,x^4}{24} + \&c.$$

6. Expand $e^{-x}\log(1+x)$ to four terms in powers of x.

$$\textit{Ans.}\quad x - \frac{3\,x^2}{2} + \frac{4\,x^3}{3} - \frac{13\,x^4}{12} + \&c.$$

7. Prove that $\log\left\{x + \sqrt{1+x^2}\right\} = x - \dfrac{x^3}{6} + \dfrac{3\,x^5}{40} - \&c.$

8. Expand $(\sec x)^n$ to three terms.

$$\textit{Ans.}\quad 1 + \frac{n\,x^2}{\underline{|2}} + \frac{(3\,n^2 + 2\,x)\,x^4}{\underline{|4}} + \&c.$$

9. Expand $e^{a+x}\cos x$ to six terms.

$$\textit{Ans.}\quad e^a\left\{1 + x - \frac{x^3}{3} - \frac{x^4}{6} - \frac{x^5}{30} + \frac{x^7}{630}\ \ldots\ \&c.\right\}.$$

10. Expand $\sin^{-1}(x+y)$, by Taylor's Theorem, to four terms in powers of y.

$$\textit{Ans.}\ \text{Sin}^{-1}x + \frac{y}{\sqrt{1-x^2}} + \frac{y^2}{\underline{|2}}\,\frac{x}{(1-x^2)^{\frac{3}{2}}} + \frac{y^3}{\underline{|3}}\,\frac{1+2\,x^2}{(1-x^2)^{\frac{5}{2}}} + \&c.$$

CHAPTER IV.

INDETERMINATE FORMS.

Art. 19.—An algebraic or transcendental function of a variable is said to be indeterminate when for a particular value of the variable the function assumes one or other of the forms $\frac{0}{0}$, $\frac{\propto}{\propto}$, 0^0, $\propto^0$ and $1^{\pm\propto}$.

For example:

$$\frac{x^2 - a\,x + c\,x - a\,c}{x^2 - a\,x + b\,x - a\,b} = \frac{0}{0} \text{ when } x = a \quad . \quad . \quad (a)$$

$$\frac{a^{\frac{1}{\frac{\pi}{2} - x}}}{\tan x} = \frac{\propto}{\propto} \text{ when } x = \frac{\pi}{2} \quad . \quad . \quad . \quad . \quad (b)$$

$$(x)^{\tan x} = 0^0 \text{ when } x = 0 \quad . \quad . \quad . \quad . \quad (c)$$

$$(\operatorname{cosec} x)^{\frac{x}{a}} = \propto^0 \text{ when } x = 0 \quad . \quad . \quad . \quad (d)$$

$$(\sin x)^{\tan x} = 1^{\propto} \text{ when } x = \frac{\pi}{2} \quad . \quad . \quad . \quad (e)$$

$$(1 + x)^{\log x} = 1^{-\propto} \text{ when } x = 0. \quad . \quad . \quad (f)$$

We shall endeavour to show how the true value of such expressions may be obtained when the variable approaches that limit which renders the expression indeterminate.

Expressions such as (*a*) and (*b*) are either in the form $\frac{f(x)}{\phi(x)}$, or they can be easily expressed in that form. Now suppose $f(x)$ and $\phi(x)$ are both zero when $x = a$.

If we substitute $(x + h)$ instead of x in each function where h is indefinitely small, we have the limiting value of

$$\frac{f(x+h)}{\phi(x+h)} = \frac{f(x)}{\phi(x)};$$

but by a theorem in algebra, each of these fractions is equal to the fraction whose numerator is the difference of the two numerators and whose denominator is the difference of the two denominators, therefore

$$\frac{f(x)}{\phi(x)} = \frac{f(x+h) - f(x)}{\phi(x+h) - \phi(x)} = \frac{\dfrac{f(x+h) - f(x)}{h}}{\dfrac{\phi(x+h) - \phi(x)}{h}},$$

that is $\frac{f^{\mathrm{I}}(x)}{\phi^{\mathrm{I}}(x)}$, therefore the limiting value of $\frac{f(x)}{\phi(x)}$ when $x = a$ is $\frac{f^{\mathrm{I}}(a)}{\phi^{\mathrm{I}}(a)}$.

If $\frac{f^{\mathrm{I}}(x)}{\phi^{\mathrm{I}}(x)}$ be indeterminate we proceed to the second derived functions, &c., until one or both functions cease to vanish or become infinite.

(72.) **Find the true value of** $\frac{x - \sin x}{\sin^3 x}$ when $x = 0$.

Here $\frac{f(x)}{\phi(x)} = \frac{x - \sin x}{\sin^3 x}$; therefore the limiting value of $\frac{x - \sin x}{\sin^3 x}$ when $x = 0$, is $\frac{f^{\mathrm{I}}(0)}{\phi^{\mathrm{I}}(0)}$; that is,

$$\frac{1 - \cos 0^0}{3 \sin^2 0^0 \cos 0^0} = \frac{1 - \cos 0^0}{3(1 - \cos^2 0^0) \cos 0^0}$$

$$= \frac{1}{3(1 + \cos 0^0) \cos 0^0} = \frac{1}{6}.$$

(73.) **Find the true value of** $\frac{\cos px - \cos qx}{\cos mx - \cos nx}$ when $x = 0$.

Here

$$\frac{f(x)}{\phi(x)} = \frac{\cos p\,x - \cos q\,x}{\cos m\,x - \cos n\,x},$$

the limiting value of which is

$$\frac{-p \sin p\,x + q \sin q\,x}{-m \sin m\,x + n \cos n\,x} = \frac{0}{0}$$

when $x = 0$, which is indeterminate. We proceed to the second derived functions, therefore the limiting value of

$$\frac{-p \sin p\,x + q \sin q\,x}{-m \sin m\,x + n \sin n\,x}$$

is

$$\frac{-p^2 \cos p\,x + q^2 \cos q\,x}{-m^2 \cos m\,x + n^2 \cos n\,x} = \frac{q^2 - p^2}{n^2 - m^2}$$

when $x = 0$.

(74.) **Find the value of** $x^2 \tan x - \left(\frac{\pi}{2}\right)^2 \sec x$ when $x = \frac{\pi}{2}$.

Here we have

$$x^2 \tan x - \left(\frac{\pi}{2}\right)^2 \sec x = \frac{x^2 \sin x - \left(\frac{\pi}{2}\right)^2}{\cos x},$$

the limiting value of which is

$$\frac{2\,x \sin x + x^2 \cos x}{-\sin x} = -\pi, \text{ when } x = \frac{\pi}{2}.$$

(75.) **Find the true value of** $(\tan x)^{\sin x}$ when $x = 0$. This is in the form 0^0.

$$\text{Let } y = (\tan x)^{\sin x}.$$

$$\therefore \quad \log y = \sin x \log \tan x = \frac{\log \tan x}{\operatorname{cosec} x},$$

the limiting value of which is

$$= -\frac{\sin x}{\cos^2 x} = 0 \text{ when } x = 0;$$

that is,

$$\log y = 0 \text{ when } x = 0.$$

$$\therefore \quad y = 1;$$

that is, the limiting value of

$$(\tan x)^{\sin x} \text{ when } x = 0, \text{ is } 1.$$

(76.) **Find the true value of** $(\tan x)^{\cos x}$ when $x = \frac{\pi}{2}$.

This is of the form $(\infty)^0$.

Let

$$y = (\tan x)^{\cos x}.$$

$$\therefore \quad \log y = \cos x \log \tan x = \frac{\log \tan x}{\sec x},$$

the limiting value of which is

$$= \frac{\cot x \sec^2 x}{\sec x \tan x} = \cos x \operatorname{cosec}^2 x = 0 \text{ when } x = \frac{\pi}{2}.$$

$$\therefore \quad (\tan x)^{\cos x} = 1 \text{ when } x = \frac{\pi}{2}.$$

(77.) **Find the true value of** $x^{\frac{1}{1-x}}$ when $x = 1$.

Let $y = x^{\frac{1}{1-x}}$. $\therefore$ $\log y = \frac{1}{1-x} \log x$, the limiting value of which is

$$= \frac{\frac{1}{x}}{-1} = -1 \text{ when } x = 1;$$

that is,

$$\log y = -1.$$

$$\therefore \quad y = e^{-1} \text{ when } x = 1.$$

EXAMPLES.

Find the limiting value of the following expressions.

1. $\dfrac{x^3 - 7x^2 + 4x + 12}{x^2 - 14x + 24}$ when $x = 2$. *Ans.* $\dfrac{6}{5}$.

2. $\dfrac{\log x}{x^4 - 1}$ when $x = 1$. *Ans.* $\dfrac{1}{4}$.

3. $\dfrac{\sin x - x^2}{x^3}$ when $x = 0$. *Ans.* ∞.

4. $\dfrac{3e^x - x^3 - 3x - 3}{\tan^2 x}$ when $x = 0$. *Ans.* $\dfrac{3}{2}$.

5. $\dfrac{\sin 2\theta + 2\sin^2\theta - 2\sin\theta}{\cos\theta - \cos^2\theta.}$ when $\theta = 0$. *Ans.* 4.

6. $\dfrac{a^x + b^x - a - b}{\log x}$ when $x = 1$. *Ans.* $\log a^a b^b$.

7. $\dfrac{a \sin x - \sin ax}{x(\cos x - \cos ax)}$ when $x = 0$. *Ans.* $\dfrac{a}{3}$.

8. $\dfrac{x^3 - 3\cos^2 x + 3}{x^2}$ when $x = 0$. *Ans.* 3.

9. $\dfrac{(e^x - e^{-x})^2}{\log(1 + x) - x}$ when $x = 0$. *Ans.* -8.

10. $\dfrac{\sin^2 4x + 2\cos^2 x - 2\cos x}{\cos^2 x - \cos^3 x}$ when $x = 0$. *Ans.* 30.

11. $\dfrac{\log(1 + x^2 + x^4)}{\sec x - \cos x}$ when $x = 0$. *Ans.* 1.

12.* $\dfrac{(x + \sin x - 4\sin\frac{1}{2}x)^4}{(3 + \cos x - 4\cos\frac{1}{2}x)^3}$ when $x = 0$. *Ans.* $\dfrac{128}{81}$.

13.* $\dfrac{(3\sin x - \sin 3x)^4}{(\sec x - \cos 2x)^6}$ when $x = 0$. *Ans.* 256.

14. $\dfrac{e^x - e^a}{\cos\dfrac{\pi x}{2a}}$ when $x = a$. *Ans.* $-\dfrac{2ae^a}{\pi}$.

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* Expand the sines and cosines by (62) and (63) before differentiating.

15. $\frac{\log x}{\sqrt{1-x}}$ when $x = 1$. *Ans.* ∞.

16. $(\cot x)^{\sin ax}$ when $x = 0$. *Ans.* 1.

17. $x^{\frac{1}{\log ax}}$ when $x = \infty$. *Ans.* e.

18. $(1 + ax)^{\frac{1}{x}}$ when $x = 0$. *Ans.* e^a.

19. $\left(\frac{A^{\frac{1}{x}} + B^{\frac{1}{x}} + C^{\frac{1}{x}} + D^{\frac{1}{x}}}{4}\right)^{4x}$ when $x = \infty$. *Ans.* $ABCD$.

20. $\frac{\cos 2x + 2\cos^2 x - 2\cos x + 1}{\sin x - \sin^2 x}$ when $x = \frac{\pi}{2}$.

Ans. ∞.

CHAPTER V.

MAXIMA AND MINIMA OF FUNCTIONS O ONE VARIABLE.

Art 20.—We have already shown in Chapter I. how t value of any function of a single variable depends upon t value we assign to the variable.

Now, suppose the variable to increase continuously fro one definite value to another, and in consequence of t gradual change, suppose the function to be gradually increasing at one time, and at another time gradually diminishing; there must be some particular value of the variable fo which the function ceases to increase and begins to diminish The corresponding value of the function is called a *maximu value.*

Again, in consequence of this gradual change of the variable, suppose that at one time the value of the function is gradually diminishing, and at another time gradually increasing; there must be some particular value of the variable for which the function ceases to diminish and begins to increase. This value is called a *minimum value of the function.*

The student should observe that the terms maximum and minimum values of a function do not necessarily mean the numerically greatest and least values of the function, as a function may have several maxima and several minima values

A maximum value of a function may be numerically less than a minimum value of it.

For a particular value of the variable, a function may cease to increase or diminish; but if it does not begin to diminish or increase respectively on passing through this value, it is neither a maximum nor a minimum.

We shall illustrate the foregoing by means of an example.

(78.) Let $y = 12x^5 - 135x^4 + 580x^3 - 1170x^2 + 1080x$. If x increases continuously from zero up to 4, on plotting the values of x horizontally and the corresponding values of y vertically, it will be found that when $x = 0$, $y = 0$, and that as x increases from 0 to 1, the value of y increases until $x = 1$. As x increases from 1 to 2, y diminishes, and when $x = 2$, y ceases to diminish. As x increases from 2 to 3, y increases, and when $x = 3$ the increment of y is zero. As x increases from 3 to 4, y increases: therefore, when $x = 1$, y is a *maximum*, and when $x = 2$, y is a *minimum*, and when $x = 3$, y is *neither* a maximum nor a minimum, since the increment of y does not change in sign as x gets greater than 3.

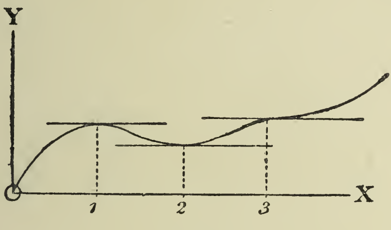


FIG. 6.

Now $\frac{dy}{dx}$ represents the rate of increase of the function, or the slope of the curve, as x increases continuously, and may be positive, zero, negative, or infinite, depending upon the instantaneous value of x. Now $\frac{dy}{dx}$ will be positive if function x is approaching a maximum value as x increases, and will be negative if function x is approaching a minimum. When function x is either a maximum or a minimum, $\frac{dy}{dx}$ is zero. This is evident on referring to the figure, since $\frac{dy}{dx}$ represents the tangent of the angle which the tangent to a curve makes with the axis of x.

Therefore we deduce the following rule for finding th values of x which make function x a maximum or minimum. *Differentiate the function of x and equate the first derived function to zero; the roots of this equation will in general render function x a maximum or minimum.*

It remains to find whether a given root renders function x a maximum, minimum, or neither.

If an adjacent value less than the given root renders $\frac{dy}{dx}$, or first derived function of $x +$, and an adjacent value greater than the root renders $\frac{dy}{dx}$, or first derived function o $x -$; then that root will render function x a *maximum*, since the function is increasing, as x increases, for an adjacent les value, and is diminishing, as x increases, for an adjacen greater value.

Similarly, if an adjacent value less than a given roo renders first derived function $x -$, and an adjacent greater value than the root renders the first derived function $x +$, then that root will render function x a *minimum*, since the function is diminishing for a smaller, and increasing for a greater value than the given root.

Again, if function x is $+$ or $-$ for an adjacent value less than a given root, and also $+$ or $-$ respectively for an adjacent value greater than the given root, that root will render the function neither a maximum nor minimum, since the first derived function does not change in sign.

(79.) If we differentiate example (78) we get

$$\frac{dy}{dx} = 60\,(x - 1)\,(x - 2)\,(x - 3)^2,$$

and equating this to zero the roots are 1, 2, and 3.

$$\begin{array}{lll} \text{If } x \text{ be} < 1 & & + \\ \quad\;\; ,, \quad\; = 1 & \dfrac{dy}{dx} & 0 \\ \quad\;\; ,, \quad\; > 1 & & - \end{array}$$

therefore $x = 1$, a maximum.

$$\begin{array}{lcc} \text{If } x \text{ be } < 2 & & - \\ \quad ,, \quad\ \ = 2 & \dfrac{d\,y}{d\,x} & 0 \\ \quad ,, \quad\ \ > 2 & & + \end{array}$$

therefore $x = 2$, a minimum.

$$\begin{array}{lcc} \text{If } x \text{ be } < 3 & & + \\ \quad ,, \quad\ \ = 3 & \dfrac{d\,y}{d\,x} & 0 \\ \quad ,, \quad\ \ > 3 & & + \end{array}$$

therefore $x = 3$, neither a maximum nor minimum.

Differentiate first derived function;

$$\therefore \quad \frac{d^2\,y}{d\,x^2} = 60\,(4\,x^3 - 27\,x^2 + 58\,x - 39).$$

Now $\frac{d^2\,y}{dx^2}$ represents the rate of change of the slope of a curve at any point on the curve; therefore, when the function is a maximum the change of slope will be negative, and when the function is a minimum the change of slope will be positive, as x gradually increases: therefore *that root which renders* $\frac{d^2\,y}{d\,x^2}$ *negative will render function* x *a maximum, and that root which renders* $\frac{d^2\,y}{d\,x^2}$ *positive will make function* x *a minimum, and that root which renders* $\frac{d^2\,y}{d\,x^2}$ *zero will in general render function* x *neither a maximum nor a minimum.*

Substituting 1, 2, and 3 for x, in $4\,x^3 - 27\,x^2 + 58\,x - 39$ it will be $-$, $+$ and 0 respectively; therefore $x = 1$ a maximum, $x = 2$ a minimum, $x = 3$ neither.

Art. 21.—In the last article we stated that a value of x which renders $\frac{d^2\,y}{d\,x^2}$ zero will in general render function x neither a maximum nor a minimum. That value of x which nders $\frac{d^2 y}{d x^2}$ zero *may* render function x a maximum or a minimum, as we shall now endeavour to show.

Let a be that value of x which renders $f(x)$ a maximum or a minimum; then

$$f(a+h) - f(a) \quad \text{and} \quad f(a-h) - f(a)$$

will both be negative if $f(a)$ be a maximum, and will both be positive if $f(a)$ be a minimum.

By **Taylor's Theorem** we have

$$f(a+h) - f(a) = h f^{\text{I}}(a) + \frac{h^2}{\underline{|2}} f^{\text{II}}(a) + \frac{h^3}{\underline{|3}} f^{\text{III}}(a) + \&\text{c.} \quad (\alpha)$$

$$f(a-h) - f(a) = -h f^{\text{I}}(a) + \frac{h^2}{\underline{|2}} f^{\text{II}}(a) - \frac{h^3}{\underline{|3}} f^{\text{III}}(a) + \&\text{c.} \quad (\beta)$$

In these equations (α) and (β), if h be very small, it is evident that the sign of the right-hand side is the same as the sign of the first term; therefore the condition for either a maximum or minimum is that $f^1(a) = 0$, for if not, then $f(a+h) - f(a)$ and $f(a-h) - f(a)$ would be opposite in sign.

$$\therefore \; f(a+h) - f(a) = \frac{h^2}{\underline{|2}} f^{\text{II}}(a) + \frac{h^3}{\underline{|3}} f^{\text{III}}(a) + \frac{h^4}{\underline{|4}} f^{\text{IV}}(a) + \&\text{c.}$$

and

$$f(a-h) - f(a) = \frac{h^2}{\underline{|2}} f^{\text{II}}(a) - \frac{h^3}{\underline{|3}} f^{\text{III}}(a) + \frac{h^4}{\underline{|4}} f^{\text{IV}}(a) - \&\text{c.}$$

If $f(a)$ be a maximum, $f(a \pm h) - f(a)$ must be negative; therefore $f^{\text{II}}(a)$ must be negative if it be not zero, since h^2 is positive. If $f(a)$ be a minimum, $f(a \pm h) - f(a)$ must be positive. Hence for a maximum or minimum $f^{\text{II}}(a)$ must be negative or positive respectively, if it be not zero.

Again, if $f^{\text{II}}(a)$ be zero, then the condition for a maximum or minimum is that $f^{\text{III}}(a) = 0$, otherwise $f(a+h) - f(a)$

and $f(a-h)-f(a)$ would be opposite in sign; there
for a maximum $f^{\text{IV}}(a)$ must be negative if not zero, an
minimum $f^{\text{IV}}(a)$ must be positive if not zero.

By similar reasoning it can be shown that if n b
the first n differential coefficients of $f(x)$ vanish w
then $f(x)$ is a maximum or minimum acco
$(n+1)$th differential coefficient is negative or p
if n be even, $f(a)$ is neither a maximum nor a mini

(80.) **To find the values of x which wil**
$y = 3x^4 - 8x^3 - 18x^2 + 72$ a maximum or minim

Here

$$\frac{dy}{dx} = 12(x^3 - 2x^2 - 3x) = 0 \text{ for a max. or mi}$$

$$\therefore \quad x = -1,\ 0, \text{ or } 3.$$

$$\frac{d^2 y}{dx^2} = 36x^2 - 48x - 36.$$

If $x = -1$ $\quad \dfrac{d^2 y}{dx^2} = +48$ positive. $\quad \therefore \quad x = -1$, a min.

If $x = 0$ $\quad \dfrac{d^2 y}{dx^2} = -36$ negative. $\quad \therefore \quad x = 0$, a max.

If $x = 3$ $\quad \dfrac{d^2 y}{dx^2} = 144$ positive. $\quad \therefore \quad x = 3$, a min.

(81.) **To find the maximum and minimum values of $y = \dfrac{x^2 - 7x + 6}{x - 10}$,**

$$\frac{dy}{dx} = \frac{(x-4)(x-16)}{(x-10)^2} = 0 \text{ for a max. or min.}$$

When $x < 4$, $= 4$, > 4, $\dfrac{dy}{dx}$ $+$, 0, $-$ respectively, therefore $x = 4$, a maximum.

When $x < 10, = 10, > 10, \frac{dy}{dx}$ $-, \propto, -$, therefore $x = 10$

er a max. nor min., since $\frac{dy}{dx}$ does not change in sign.

$< 16, = 16, > 16, \frac{dy}{dx}$ $-, 0, +$ respectively, there-

, a minimum.

$= 4$, $y = 1$, a maximum. If $x = 16$, $y = 25$, a

m.

this example the maximum is less than the minimum

Art. 20).

It would be interesting to plot on squared paper the curve epresenting the relation between y and x, observing what occurs as x gradually increases from 0 up to 20.

(82.) **Divide a line into two parts such that the rectangle under them may be a maximum.**

Let a denote the line. Then, if x denote one part, $a - x$ will denote the other part, and $x(a - x)$ is to be a maximum.

Let $y = x(a - x)$. Therefore $\frac{dy}{dx} = a - 2x = 0$ for a maximum.

$$\therefore \quad x = \frac{a}{2},$$ that is, the line must be bisected.

(83.) **Find two factors of a, so that the sum of their squares is a minimum.**

Let x denote one factor, then $\frac{a}{x}$ denotes the other factor.

Therefore $y = x^2 + \frac{a^2}{x^2}$ is to be a minimum.

$$\frac{dy}{dx} = 2x - \frac{2a^2}{x^3} = 0 \text{ for a minimum.} \qquad \therefore \quad x = \sqrt{a}.$$

(84.) **The strength of a rectangular beam of given length and material and loaded in any particular way, is proportional to its breadth and to the square of its depth. What is the breadth of the**

strongest beam that can be cut from a cylindr tree of 12 inches diameter?

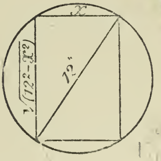


FIG. 7.

Let x denote the required breadth in inches; then $\sqrt{12^2 - x^2}$ will denote the depth. The strength is proportional to $x\,(12^2 - x^2)$.

Let $y = x\,(12^2 - x^2)$.

$$\therefore \quad \frac{d\,y}{d\,x} = 144 - 3\,x^2 = 0 \text{ for a maximum.}$$

$$\therefore \quad x = 4\,\sqrt{3} \text{ inches, and depth} = 4\sqrt{6} \text{ inches.}$$

(85.) **To find the depth of the stiffest beam that can be cut from a cylindric tree 12 inches in diameter and of given length. Stiffness is proportional to the breadth and to the cube of the depth.**

Let x denote the required breadth; then $\sqrt{12^2 - x^2}$ will denote the depth in inches.

The stiffness is proportional to $x\,(12^2 - x^2)^{\frac{3}{2}}$.

Let $y = x\,(12^2 - x^2)^{\frac{3}{2}}$.

$$\therefore \quad \frac{d\,y}{d\,x} = (12^2 - x^2)^{\frac{3}{2}} - 3\,x^2\,(12^2 - x^2)^{\frac{1}{2}} = 0 \text{ for a maximum.}$$

Therefore $144 - 4\,x^2 = 0$. $\therefore$ $x = 6$ inches, the required breadth; therefore the required depth is $6\,\sqrt{3}$ inches.

(86.) **If $C^2 r + \frac{t^2}{r}$ is the total waste per mile going on in an electric conductor (r ohms resistance per mile), due to heat, interest and depreciation, find the relation between C, r and t when the waste is a minimum.**

Let $y = C^2 r + \frac{t^2}{r}$, C and t being constants.

$$\frac{d\,y}{d\,r} = C^2 - \frac{t^2}{r^2} = 0 \text{ for a minimum.}$$

Therefor[illegible] $r = t$.

Lord Kelvin's rule is $C r = 17$, where C is the current in amperes, r the resistance in ohms per mile of conductor. The numerical value of r in terms of the cross-sectional area a of conductor is $\frac{\cdot 04}{a}$, approximately.

$$\therefore \quad a = \frac{C}{425}.$$

For a minimum cost the carriage is therefore 425 amperes per square inch of cross-sectional area of conductor.

(87.) **To find the proper section of a conductor to transmit a given power of P watts over a distance of n miles, taking into account the drop in potential due to the distance.**

Let $V_{\prime}$ be the potential at the generator, and V the potential at the motor, r the resistance in ohms per mile, therefore the total resistance is $n r$. $V_{\prime} - V =$ drop in volts $= n r C$, where C is the current in amperes; therefore

$$V = V_{\prime} - n r C, \quad \text{and } P = C(V_{\prime} - n r C);$$

therefore

$$r = \frac{C V_{\prime} - P}{n C^2}.$$

Substituting this value of r in the rule for waste,

$$C^2 r + \frac{t^2}{r},$$

we have waste

$$W = \frac{C^2 (C V_{\prime} - P)}{n C^2} + \frac{t^2 n C^2}{C V_{\prime} - P}.$$

We want the waste to be a minimum, C being the independent variable, $V_{\prime}$, P, n, and t being constants,

$$\therefore \quad \frac{dW}{dC} = \frac{V_{\prime}}{n} + \frac{2 n t^2 C (C V_{\prime} - P) - V_{\prime} t^2 n C^2}{(C V_{\prime} - P)^2} = 0$$

for a minimum, therefore

$$V_{\prime} (C V_{\prime} - P)^2 + 2 n^2 t^2 C (C V_{\prime} - P) - V_{\prime} t^2 n^2 C^2 = 0. \qquad (a)$$

14. Find the cone of maximum volume that can scribed in a hemisphere of radius r, its vertex being at the centre of the sphere.

Ans. $\text{Vol} = \dfrac{2\sqrt{3}\,\pi r^3}{27}$.

15. Find the cone of minimum volume that can be described about a given hemisphere.

Ans. $\text{Vol} = \dfrac{\pi r^3 \sqrt{3}}{2}$.

16. The area of a rectangle is given; find the ratio of the lengths of two adjacent sides if the sum of three sides is a minimum.

Ans. 2 : 1.

17. Find the cone of minimum surface that can be described about a given hemisphere.

Ans. $\text{Vol} = \dfrac{8\pi r^3}{9}$.

18. Find the height of the flame of a lamp standing on the centre of a round table 4 ft. in diameter, so that a given horizontal area at its edge may receive the greatest illumination from it. The intensity of light varies directly as the sine of the angle which a ray makes with the plane of the table, and inversely as the square of the distance.

Ans. Height $= 12\sqrt{2}$ inches.

19. Find the greatest cylinder that can be inscribed in a given hemisphere.

Ans. $\text{Vol} = \dfrac{2\sqrt{3}\,\pi r^3}{9}$.

20. A line A B is terminated by the axes of X and Y and passes through a fixed point P, whose distances from the axes

of X and Y are 5 and 3 respectively; find its direction when $A P^2 + B P^2$ is a minimum.

Ans. $\text{Tan}^{-1} - \sqrt{\frac{5}{3}}$ with axis of X.

21. Find the dimensions of the strongest rectangular beam of given length that can be cut from a cylindric tree of a feet in diameter.

Ans. Breadth $= \frac{a\sqrt{3}}{3}$.

Depth $= \frac{a\sqrt{6}}{3}$.

22. Find the stiffest beam that can be cut from the tree mentioned in Example 21.

Ans. Depth $= \frac{a\sqrt{3}}{2}$.

Breadth $= \frac{a}{2}$.

23. Find the proper section of a conductor to transmit 60,000 watts over a distance of 10 miles, the potential at the generator being 2000 volts. (See Example 87.)

Ans. ·076 sq. inch.

24. At what distance from the wall of a house must a man, whose eye is $5\frac{1}{2}$ feet above the ground, station himself, in order that a window 5 feet high, whose sill is $20\frac{1}{2}$ feet above the ground, may subtend the greatest vertical angle.

Ans. $10\sqrt{3}$ feet.

25. Given the length of an arc of a circle l, find what portion of a circle it must be so that the corresponding segment shall be a maximum.

Ans. $l = \pi r$, where r is the radius of the circle.

CHAPTER VI.

DIFFERENTIAL COEFFICIENT OF A FUNCTION OF TWO OR MORE VARIABLES, AND OF IMPLICIT FUNCTIONS.

ART 22.—Let $v = f(x, y)$ where x and y are both variables, and suppose x and y to receive finite increments δx and δy respectively, and let δv denote the corresponding increment of v. We have

$$v + \delta v = f(x + \delta x,\ y + \delta y).$$

$$\therefore\ \delta v = f(x + \delta x,\ y + \delta y) - f(x,\ y)$$

$$= f(x + \delta x, y + \delta y) - f(x, y + \delta y) + f(x, y + \delta y) - f(x, y)$$

$$= \frac{\{f(x + \delta x,\ y + \delta y) - f(x,\ y + \delta y)\}}{\delta x}\,\delta x$$

$$+ \frac{\{f(x,\ y + \delta y) - f(x,\ y)\}}{\delta y}\,\delta y.$$

Suppose δx and δy, and consequently δv, to diminish indefinitely; therefore

$$d v = \frac{\{f(x + d x,\ y + d y) - f(x,\ y + d y)\}}{d x}\,d x$$

$$+ \frac{\{f(x, y + d y) - f(x, y)\}}{d y}\,d y,$$

that is,

$$d v = \left(\frac{d v}{d x}\right) d x + \left(\frac{d v}{d y}\right) d y, \quad . \quad . \quad . \ (a)$$

where

$$\left(\frac{d\,v}{d\,x}\right) = \frac{f(x + d\,x,\ y + d\,y) - f(x,\ y + d\,y)}{d\,x}$$

$$= \frac{f(x + d\,x,\ y) - f(x\,y)}{d\,x},$$

when $d\,y$ is indefinitely small. That is, $\left(\frac{d\,v}{d\,x}\right)$ is the differential coefficient of $f(x,\ y)$, treating y as a constant, and

$$\left(\frac{d\,v}{d\,y}\right) = \frac{f(x,\ y + dy) - f(x,\ y)}{d\,y},$$

that is, the differential coefficient of $f(x,\ y)$, treating x as a constant.

$\left(\frac{d\,v}{d\,x}\right)$ and $\left(\frac{d\,v}{d\,y}\right)$ are called the partial differential coefficients of $f(x,\ y)$ with respect to x and y respectively, and $d\,v$ is the complete differential of $f(x,\ y)$, when x and y both vary. Therefore the complete differential of a function of two variables is the sum of the differentials of the function with respect to the two variables.

(93.) Let

$$v = a\,x^2 + 2\,b\,x\,y + c\,y^2\,;$$

therefore

$$d\,v = 2\,(a\,x + b\,y)\,d\,x + 2\,(b\,x + c\,y)\,d\,y.$$

Here

$$\left(\frac{d\,v}{d\,x}\right) d\,x = 2\,(a\,x + b\,y)\,d\,x$$

and

$$\left(\frac{d\,v}{d\,y}\right) d\,y = 2\,(b\,x + c\,y)\,d\,y.$$

(94.) Let

$$v = x \sin y\,;$$

therefore

$$d\,v = \sin y\,d\,x + x \cos y\,d\,y.$$

(95.) Let

$$v = x^n y^m;$$

therefore

$$d v = n x^{n-1} y^m \, d x + m y^{m-1} x^n d y.$$

ART. 23.—If $v = f(x, y, z)$, and if x, y and z receive finite increments, δx, δy and δz, and in consequence let v become $v + \delta v$; therefore

$$v + \delta v = f(x + \delta x,\ y + \delta y,\ z + \delta z);$$

therefore

$$\begin{aligned}\delta v &= f(x + \delta x,\ y + \delta y,\ z + \delta z) - f(x,\ y,\ z)\\ &= f(x + \delta x,\ y + \delta y,\ z + \delta z) - f(x,\ y + \delta y,\ z + \delta z)\\ &\quad + f(x,\ y + \delta y,\ z + \delta z) - f(x,\ y,\ z + \delta z)\\ &\quad + f(x,\ y,\ z + \delta z) - f(x,\ y,\ z)\\ &= \frac{\{f(x + \delta x,\ y + \delta y,\ z + \delta z) - f(x,\ y + \delta y,\ z + \delta z\}}{d x} d x\\ &\quad + \frac{\{f(x,\ y + \delta y,\ z + \delta z) - f(x,\ y,\ z + \delta z)\}}{d y} d y\\ &\quad + \frac{\{f(x,\ y,\ z + \delta z) - f(x,\ y,\ z)}{d z} d z,\end{aligned}$$

that is,

$$d v = \left(\frac{d v}{d x}\right) d x + \left(\frac{d v}{d y}\right) d y + \left(\frac{d v}{d z}\right) d z,$$

when δx, δy and δz become indefinitely small.

Similarly, we could extend this process of reasoning to a function of any number of variables.

Let $v = f(x, y, z, \ldots.)$, therefore,

$$d v = \left(\frac{d v}{d x}\right) d x + \left(\frac{d v}{d y}\right) d y + \left(\frac{d v}{d z}\right) d z. \ \ldots .$$

(96.) Let

$$v = \frac{x^2}{a^2} + \frac{y^2}{h^2} + \frac{z^2}{c^2},$$

$$\therefore \quad d v = \left(\frac{2 x}{a^2}\right) d x + \left(\frac{2 y}{h^2}\right) d y + \left(\frac{2 z}{c^2}\right) d z.$$

ART. 24.—In the equation (a) of Art. 22, if $v = o$, we have $f(x, y) = 0$, and we say that x is an implicit function of y, or y is an implicit function of x. Since $f(x, y) = 0$ for all values of x and y, therefore,

$$f(x + \delta x, y + \delta y) = 0, \text{ hence } d v = 0,$$

therefore

$$\left(\frac{d v}{d x}\right) d x + \left(\frac{d v}{d y}\right) d y = 0,$$

therefore

$$\frac{d y}{d x} = - \frac{\left(\frac{d v}{d x}\right)}{\left(\frac{d v}{d y}\right)}.$$

(97.) Find $\frac{d y}{d x}$, given $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$.

Here

$$\frac{d v}{d x} = \frac{2 x}{a^2}, \quad \text{and} \quad \frac{d v}{d y} = \frac{2 y}{b^2},$$

therefore

$$\frac{d y}{d x} = - \frac{b^2}{a^2} \frac{x}{y}.$$

(98.) To find $\frac{d v}{d x}$ where $v = y^3 + z^3 + 3 a y z$, and $y = \tan x$ $z = a^x$.

Here

$$v = f(y, z). \qquad \therefore \quad d v = \left(\frac{d v}{d y}\right) d y + \left(\frac{d v}{d z}\right) d z,$$

and since y and z are functions of x,

$$\frac{d v}{d x} = \left(\frac{d v}{d y}\right) \frac{d y}{d x} + \left(\frac{d v}{d z}\right) \frac{d z}{d x}. \quad . \; . \; . \; . \; (a)$$

Now

$$\left(\frac{d v}{d y}\right) = 3 (y^2 + a z), \quad \frac{d y}{d x} = \sec^2 x, \quad \left(\frac{d v}{d z}\right) = 3 (z^2 + a y),$$

and

$$\frac{dz}{dx} = a^x \log a.$$

On substituting in (a) we get

$$\frac{dv}{dx} = 3\,\{(y^2 + a z) \sec^2 x + (z^2 + a y)\, a^x \log a\}.$$

$$\therefore \frac{dv}{dx} = 3\,\{(\tan^2 x + a^{x+1}) \sec^2 x + (a^{2x} + a \tan x)\, a^x \log a\}.$$

(99.) Let $v = \tan^{-1}\left(\frac{z}{y}\right)$, where $z = f(x)$, and $y = \phi(x)$, find $\frac{dv}{dx}$.

$$\frac{dv}{dz} = \frac{y}{y^2 + z^2}, \quad \text{and} \quad \frac{dv}{dy} = -\frac{z}{y^2 + z^2}.$$

Substituting these values in (a) of Example (98) we get

$$\frac{dv}{dx} = \frac{1}{y^2 + z^2}\,\{y f'(x) - z \phi'(x)\}.$$

(100.) If $v = \frac{z^2 - y^2}{z^2 + y^2}$, find $\frac{dv}{dx}$, where y and z are functions of x.

We have

$$\frac{dv}{dx} = \left(\frac{dv}{dz}\right)\frac{dz}{dx} + \left(\frac{dv}{dy}\right)\frac{dy}{dx}. \quad \text{See 98.}$$

Here

$$\frac{dv}{dz} = \frac{4 y^2 z}{(z^2 + y^2)^2}, \quad \text{and} \quad \frac{dv}{dy} = \frac{-4 y z^2}{(z^2 + y^2)^2}.$$

$$\therefore \quad \frac{dv}{dx} = \frac{4 y^2 z}{(z^2 + y^2)^2}\frac{dz}{dx} - \frac{4 y z^2}{(z^2 + y^2)^2}\frac{dy}{dx}$$

$$= \frac{4 y z}{(z^2 + y^2)^2}\left\{y \phi'(x) - z f'(x)\right\},$$

here $y = f(x)$ and $z = \phi(x)$.

(101.) Given $x^{ay} - y^{bx} = 0$, find $\dfrac{dy}{dx}$

Here

$$a y \log x = b x \log y;$$

therefore

$$a \frac{dy}{dx} \log x + \frac{ay}{x} = b \log y + \frac{bx}{y} \frac{dy}{dx}.$$

$$\therefore \quad \frac{dy}{dx}\left(a \log x - \frac{bx}{y}\right) = b \log y - \frac{ay}{x}.$$

$$\therefore \quad \frac{dy}{dx} = \frac{bxy \log y - ay^2}{axy \log x - bx^2}.$$

EXAMPLES.

1. If $v = \dfrac{z - y}{z + y}$, prove that

$$dv = \frac{2(y\,dz - z\,dy)}{(y + z)^2}.$$

2. If $v = \dfrac{x^a}{y^b}$, prove that

$$dv = \frac{x^{a-1}}{y^{b+1}}\left(ay\,dx - bx\,dy\right).$$

3. If $v = \tan^{-1}\left(\dfrac{z}{y}\right)$, prove that

$$dv = \frac{y\,dz - z\,dy}{y^2 + z^2}.$$

4. If $v = a^{xyz}$, prove that

$$dv = a^{xyz} \log a\,(xy\,dz + xz\,dy + yz\,dx).$$

5. If $v = a^{x+y+z}$, prove that

$$dv = a^{x+y+z} \log a\,(dx + dy + dz).$$

6. If $v = \sec^{-1}\left(\dfrac{z}{y}\right)$ where $y = \phi(x)$ and $z = f(x)$, prove that

$$\frac{dv}{dx} = \frac{1}{\sqrt{(z^2 - y^2)}}\left\{\frac{\phi(x)\,f'(x) - f(x)\,\phi'(x)}{f(x)}\right\}.$$

W1

7. If $v = \sqrt{\frac{x^2 - y^2}{x^2 + y^2}}$, prove that

$$d\,v = \frac{2\,x\,y\,(y\,d\,x - x\,d\,y)}{(x^2 + y^2)^{\frac{3}{2}}\,\sqrt{(x^2 - y^2)}}.$$

8. If $v = \frac{x^2\,y}{a^2 - z^2}$, prove that

$$d\,v = \frac{2\,x\,y\,d\,x + x^2\,d\,y}{a^2 - z^2} + \frac{2\,x^2\,y\,z\,d\,z}{(a^2 - z^2)^2}.$$

9. If $v = \log \tan \left(\frac{x}{y}\right)$, prove that

$$d\,v = \frac{2\,y\,d\,x - 2\,x\,d\,y}{y^2 \sin \frac{2\,x}{y}}.$$

10. If $u = x^3 + y^3 + 3\,x^2\,y + 4\,x\,y^2$, prove that

$$\frac{d^2\,u}{d\,y\,d\,x} = \frac{d^2\,u}{d\,x\,d\,y}.$$

11. Given $u = a\,x^3 + b\,x^2\,y\,z + c\,x\,y^2 + d\,z^3$, find

$$\frac{d\,u}{d\,x},\ \frac{d\,u}{d\,y},\ \frac{d\,u}{d\,z}.$$

$$\frac{d\,u}{d\,x} = 3\,a\,x^2 + 2\,b\,x\,y\,z + cy,^2$$

$$\frac{d\,u}{d\,y} = b\,x^2\,z + 2\,c\,x\,y,$$

$$\frac{d\,u}{d\,z} = b\,x^2\,y + 3\,d\,z^2.$$

CHAPTER VII.

ART. 25.—Maxima and Minima of Functions of Two or Three Variables.

BY an extension of Taylor's Theorem it can be proved that

$$f(x+\alpha, y+\beta) - f(x, y) = \alpha \frac{d f}{d x} + \beta \frac{d f}{d y}$$

$$+ \frac{}{\underline{|2}} \left\{ \alpha^2 \frac{d^2 f}{d x^2} + 2 \alpha \beta \frac{d^2 f}{d x \, d y} + \beta^2 \frac{d^2 f}{d y^2} \right\}$$

$$+ \frac{1}{\underline{|3}} \left\{ \alpha^3 \frac{d^3 f}{d x^3} + 3 \alpha^2 \beta \frac{d^3 f}{d x^2 \, d y} + 3 \alpha \beta^2 \frac{d^3 f}{d x \, d y^2} + \beta^3 \frac{d^3 f}{d y^3} \right\}$$

$$+ \&c. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(A)}$$

Space does not permit us to give the proof of this expa sion, and we therefore refer the reader to more advanc treatises on the subject.

Now, $\frac{d^2 f}{d x \, d y}$ means that $f(x, y)$ is differentiated, first w regard to y treating x as constant, and the result different ated with regard to x, treating y as constant. $\frac{d^3 f}{d x^2 \, d y}$ mean that $f(x, y)$ is differentiated first with regard to y treating as constant, and the result differentiated twice with regar to x treating y as constant, &c. If $f(x, y)$ be a maximum or a minimum when $x = a$ and $y = b$, therefore by similar reasoning to that given in Art. 21, it follows that

$$f(a \pm \alpha, b \pm \beta) - f(a, b)$$

must be negative or positive respectively; and when α and β are very small, the sign of the right-hand side of A is the same as the sign of

$$\alpha \frac{d f}{d x} + \beta \frac{d f}{d y};$$

therefore

$$\frac{d f}{d x} = 0 \quad \text{and} \quad \frac{d f}{d y} = 0,$$

otherwise

$$f(a + \alpha, b + \beta) - f(a, b)$$

and

$$f(a - \alpha, b - \beta) - f(a, b)$$

would have opposite signs. Again, for a maximum or a minimum

$$\alpha^2 \frac{d^2 f}{d x^2} + 2 \alpha \beta \frac{d^2 f}{d x\, d y} + \beta^2 \frac{d^2 f}{d y^2} \quad . \quad . \quad (B)$$

must be negative or positive respectively, if not zero, for very small values of α and β.

Denoting the differential coefficients $\frac{d^2 f}{d x^2}$, $\frac{d^2 f}{d x\, d y}$ and $\frac{d^2 f}{d y^2}$ by X, Y and Z respectively, we have

$$\alpha^2 X + 2 \alpha \beta Y + \beta^2 Z \text{ instead of (B)},$$

that is,

$$\frac{\beta^2}{X}\left\{\left(X \frac{\alpha}{\beta} + Y\right)^2 + X Z - Y^2\right\}.$$

In order that this expression may be always positive or egative for small values of α and β, it is obvious that $X Z - Y^2$ must not be negative; therefore the condition for a real aximum or minimum is that $\frac{d^2 f}{d x^2} \times \frac{d^2 f}{d y^2}$ must be greater than or equal to $\left(\frac{d^2 f}{d x\, d y}\right)^2$.

If this condition be satisfied, $f(x, y)$ will be a maximum or minimum, according as X, that is $\frac{d^2 f}{d x^2}$, is negative or positive; since $\left(X\frac{\alpha}{\beta}+Y\right)^2$ is always positive and also β^2 is positive.

Therefore we must have the following conditions satisfied, if $f(x\,y)$ be a maximum or minimum.

$$\frac{d\,f}{d\,x}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{d\,f}{d\,y}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\frac{d^2 f}{d\,x^2}=\text{negative quantity for a } \textit{maximum} \quad . \quad (3)$$

$$\frac{d^2 f}{d\,x^2}=\text{positive quantity for a } \textit{minimum} \quad . \quad (4)$$

and

$$\frac{d^2 f}{d\,x^2}\times\frac{d^2 f}{d\,y^2}-\left(\frac{d^2 f}{d\,x\,d\,y}\right)^2$$

must not be a negative quantity. . . . . (5)

If $\left(\frac{d^2 f}{d\,x^2}\,\frac{d^2 f}{d\,y^2}\right)-\left(\frac{d^2 f}{d\,x\,d\,y}\right)^2$ be negative, then $f(x,\ y)$ will be neither a maximum nor a minimum.

(102.) **To find the maximum or minimum value of** $v=x^2+y^2-4\,a\,x-6\,b\,y$.

Here

$$\frac{d\,v}{d\,x}=2\,x-4\,a=0$$

and

$$\frac{d\,v}{d\,y}=2\,y-6\,b=0$$

for a maximum or minimum. Therefore

$$x=2\,a,\text{ and } y=3\,b.$$
$$\therefore\quad v=-(4\,a^2+9\,b^2).$$

$$\frac{d^2\,v}{d\,x^2}=2,\text{ a positive quantity, and }\frac{d^2\,v}{d\,x\,d\,y}=0.$$

Therefore $v=-(4\,a^2+9\,b^2)$ is a minimum value.

(103.) **To find the point the sum of the squares of whose distances from two given points (a_1, b_1) and (a_2, b_2) may be the least possible.**

The result is evident, but we give it as an illustration of the rules employed.

Let (x, y) be the required point; then

$$u = (x - a_1)^2 + (y - b_1)^2 + (x - a_2)^2 + (y - b_2)^2$$

is to be a minimum

$$\left.\begin{array}{l} \dfrac{du}{dx} = 2x - a_1 - a_2 = 0 \\[2ex] \text{and} \\[2ex] \dfrac{du}{dy} = 2y - b_1 - b_2 = 0 \end{array}\right\} \text{for a minimum.}$$

Also

$$\frac{d^2 u}{dx^2} = 2 \text{ a positive quantity,}$$

and

$$\frac{d^2 u}{dx\,dy} = 0.$$

Therefore u is a minimum, and the co-ordinates of the required point are

$$x = \frac{a_1 + a_2}{2}, \quad y = \frac{b_1 + b_2}{2},$$

that is, the middle point of the straight line joining (a_1, b_1) and (a_2, b_2).

Art. 26.—**Maxima and Minima of Three Variables.**

If $f(x + \alpha, y + \beta, z + \gamma)$ be expanded in powers of α, β and γ, by an extension of Taylor's Theorem, we have

$$f(x + \alpha, y + \beta, z + \gamma) - f(x, y, z) = \alpha \frac{df}{dx} + \beta \frac{df}{dy} + \gamma \frac{df}{dy}$$

$$+ \frac{\alpha^2}{2} \frac{d^2 f}{dx^2} + \frac{\beta^2}{2} \frac{d^2 f}{dy^2} + \frac{\gamma^2}{2} \frac{d^2 f}{dz^2} + \alpha \beta \frac{d^2 f}{dx\,dy} + \alpha \gamma \frac{d^2 f}{dx\,dz}$$

$$+ \beta \gamma \frac{d^2 f}{dy\,dz} + \&c. \quad . \quad . \quad . \quad . \quad \text{(A)}$$

By taking α, β, γ small enough, the sign of the right-hand side of A will depend upon that of the terms involving only the first powers of α, β and γ; therefore for a maximum or minimum,

$$\alpha \frac{d f}{d x} + \beta \frac{d f}{d y} + \gamma \frac{d f}{d z} = 0,$$

and since α, β and γ are independent of each other, it follows that

$$\frac{d f}{d x} = 0, \quad \frac{d f}{d y} = 0, \quad \frac{d f}{d z} = 0. \quad . \quad . \quad (B)$$

These three equations will determine the values of x, y and z, which will make $f\ (x, y, z)$ a maximum or minimum, if such exist. By reasoning similar to that given in the last article, the conditions for a maximum or minimum are that

$$\mathrm{P\,Q} > \mathrm{R}'^2, \text{ and}$$

$$\mathrm{P}\,\{\mathrm{P\,Q\,R} + 2\,\mathrm{P}'\,\mathrm{Q}'\,\mathrm{R}' - \mathrm{P\,P}'^2 - \mathrm{Q\,Q}'^2 - \mathrm{R\,R}'^2\} > 0$$

where

$$\mathrm{P} = \frac{d^2 f}{d x^2}, \quad \mathrm{Q} = \frac{d^2 f}{d y^2}, \quad \mathrm{R} = \frac{d^2 f}{d z^2},$$

$$\mathrm{P}' = \frac{d^2 f}{d y\, d z}, \quad \mathrm{Q}' = \frac{d^2 f}{d x\, d z}, \quad \mathrm{R}' = \frac{d^2 f}{d x\, d y}.$$

For a maximum $\frac{d^2 f}{d x^2}$ must be negative.

„ minimum „ „ positive.

Art. 27.—Undetermined Multipliers.

Let $v = f\,(x, y, z)$ be a function of three variables, x, y, z, which are connected by the equation $\phi\ (x, y, z) = 0$, it is required to find a maximum or minimum value of v.

For a maximum or minimum

$$\left(\frac{d v}{d x}\right) d x + \left(\frac{d v}{d y}\right) d y + \left(\frac{d v}{d z}\right) d z = 0\,. \quad . \quad (A)$$

d from the equation $\phi(x, y, z) = 0$, we have

$$\left(\frac{d\phi}{dx}\right)dx + \left(\frac{d\phi}{dy}\right)dy + \left(\frac{d\phi}{dz}\right)dz = 0. \quad . \quad \text{(B)}$$

Multiply equation B, by λ, an arbitrary constant, and add result to A; thus

$$\left\{\left(\frac{dv}{dx}\right) + \lambda\left(\frac{d\phi}{dx}\right)\right\}dx + \left\{\left(\frac{dv}{dy}\right) + \lambda\left(\frac{d\phi}{dy}\right)\right\}dy + \left\{\left(\frac{dv}{dz}\right) + \lambda\left(\frac{d\phi}{dz}\right)\right\}dz = 0. \quad . \quad . \quad . \quad . \quad \text{(C)}$$

Since λ is an arbitrary constant, we may give to it such lue that

$$\left(\frac{dv}{dx}\right) + \lambda\left(\frac{d\phi}{dx}\right) = 0,$$

efore C becomes

$$\left\{\left(\frac{dv}{dy}\right) + \lambda\left(\frac{d\phi}{dy}\right)\right\}dy + \left\{\left(\frac{dv}{dz}\right) + \lambda\left(\frac{d\phi}{dz}\right)\right\}dz = 0, \quad \text{(D)}$$

since y and z may be considered independent variables, use y and z can be expressed in terms of x by means of tions $v = f(x, y, z)$ and $\phi(x, y, z) = 0$, it follows that

$$\left\{\left(\frac{dv}{dy}\right) + \lambda\left(\frac{d\phi}{dy}\right)\right\}dy = 0 \quad . \quad . \quad . \quad \text{(E)}$$

$$\left\{\left(\frac{dv}{dz}\right) + \lambda\left(\frac{d\phi}{dz}\right)\right\}dz = 0. \quad . \quad . \quad . \quad \text{(F)}$$

fore we have the three equations

$$\left(\frac{dv}{dx}\right) + \lambda\left(\frac{d\phi}{dx}\right) = 0$$

$$\left(\frac{dv}{dy}\right) + \lambda\left(\frac{d\phi}{dy}\right) = 0$$

$$\left(\frac{dv}{dz}\right) + \lambda\left(\frac{d\phi}{dz}\right) = 0,$$

together with $\phi(x, y, z) = 0$ for determining the values
x, y, z and λ.

(104.) **To find the rectangular parallelopiped maximum surface that can be inscribed in a sph whose equation is**

$$x^2 + y^2 + z^2 = r^2. \quad . \quad . \quad . \quad . \quad (1$$

The surface is

$$S = 8\,(x\,y + x\,z + y\,z), \quad . \quad . \quad . \quad (2$$

where $2\,x$, $2\,y$ and $2\,z$ are the lengths of its three cotermin edges.

By equations (1) and (2) we have

$$x\,d\,x + y\,d\,y + z\,d\,z = 0$$

and

$$(y + z)\,d\,x + (x + z)\,d\,y + (y + x)\,d\,z = 0$$

for a maximum; therefore

$$y + z + \lambda\,x = 0 \quad . \quad . \quad . \quad . \quad ($$
$$x + z + \lambda\,y = 0 \quad . \quad . \quad . \quad .$$
$$x + y + \lambda\,z = 0. \quad . \quad . \quad . \quad .$$

Multiplying (a), (b) and (c) by x, y and z respect and adding, we get

$$2\,(x\,y + x\,z + y\,z) + \lambda\,(x^2 + y^2 + z^2) = 0,$$

that is,

$$\frac{S}{4} + \lambda\,r^2 = 0.$$

This determines λ, and by means of equations and (c) it is evident that $x = y = z$; therefore the re

rectangular parallelopiped is a cube whose edge is $2\,x =$

by equation (1).

(105.) **To find the volume of the greatest angular block that can be inscribed in the elli whose equation is**

$$\frac{x^2}{l^2} + \frac{y^2}{m^2} + \frac{z^2}{n^2} = 1. \quad . \quad . \quad . \quad .$$

$2x$, $2y$ and $2z$ denote the lengths of three coternous edges of the block; then its volume

$$V = 8xyz. \quad . \quad . \quad . \quad . \quad . \quad (b)$$

From (a) and (b) we get by differentiating

$$\frac{2x\,dx}{l^2} + \frac{2y\,dy}{m^2} + \frac{2z\,dz}{n^2} = 0$$

and

$$yz\,dx + xz\,dy + xy\,dz = 0$$

for a maximum; therefore

$$yz + \lambda\frac{x}{l^2} = 0, \quad xz + \lambda\frac{y}{m^2} = 0$$

and

$$xy + \lambda\frac{z}{n^2} = 0.$$

Multiplying these three equations by x, y and z respectively, we have

$$xyz + \lambda\frac{x^2}{l^2} = 0. \quad . \quad . \quad . \quad . \quad (a)$$

$$xyz + \lambda\frac{y^2}{m^2} = 0. \quad . \quad . \quad . \quad . \quad (b)$$

$$xyz + \lambda\frac{z^2}{n^2} = 0, \quad . \quad . \quad . \quad . \quad (c)$$

n adding (a), (b) and (c) we have

$$3xyz + \lambda\left(\frac{x^2}{l^2} + \frac{y^2}{m^2} + \frac{z^2}{n^2}\right) = 0;$$

is,

$$\frac{3V}{8} + \lambda = 0;$$

efore

$$\lambda = -\frac{3V}{8}.$$

Substituting $-\frac{3V}{8}$ for λ in (a), (b) and (c) we get

$$2x = \frac{2l}{\sqrt{3}}, \quad 2y = \frac{2m}{\sqrt{3}} \quad \text{and} \quad 2z = \frac{2n}{\sqrt{3}};$$

therefore the maximum volume is

$$V = \frac{8lmn}{3\sqrt{3}}.$$

(106.) **To find the minimum value of**

$$u = ax^2 + by^2 + cz^2$$

subject to the conditions

$$Ax + By + Cz = D.$$

Here

$$du = 2ax\,dx + 2by\,dy + 2cz\,dz = 0$$

for a maximum, and

$$A\,dx + B\,dy + C\,dz = 0;$$

therefore

$$2ax + \lambda A = 0, \quad 2by + \lambda B = 0, \quad 2cz + \lambda C =$$

Hence

$$2(ax^2 + by^2 + cz^2) + \lambda(Ax + By + Cz) = 0$$

that is,

$$2u + \lambda D = 0$$

$$\therefore \quad \lambda = -\frac{2u}{D}.$$

Substituting for λ in

$$2ax + \lambda A = 0, \text{ \&c.},$$

we get

$$2ax = \frac{2uA}{D},$$

$$\therefore \quad x = \frac{u\,A}{a\,D},$$

$$\therefore \quad a\,x^2 = \frac{u^2\,A^2}{a\,D^2}.$$

Similarly, we get

$$b\,y^2 = \frac{u^2\,B^2}{b\,D^2},$$

and

$$c\,z^2 = \frac{u^2\,C^2}{c\,D^2},$$

and on adding we have

$$a\,x^2 + b\,y^2 + c\,z^2 = \frac{u^2}{D^2}\left(\frac{A^2}{a} + \frac{B^2}{b} + \frac{C^2}{c}\right) = u.$$

$$\therefore u = \frac{D^2}{\frac{A^2}{a} + \frac{B^2}{b} + \frac{C^2}{c}}.$$

(107.) To find the dimensions of a cistern of maximum capacity that can be formed out of 300 square eet of sheet iron, there being no lid.

Let x denote the required length, y its breadth and z its epth, all in feet. Then

d

$$x\,y + 2\,x\,z + 2\,y\,z = 300 = A, \text{ say},$$

d $u = x\,y\,z$ is to be a maximum.

From these equations we get

$$(y + 2\,z)\,d\,x + (x + 2\,z)\,d\,y + (2\,x + 2\,y)\,d\,z = 0,$$

d

$$y\,z\,d\,x + x\,z\,d\,y + x\,y\,d\,z = 0$$

r a maximum. Hence

$$y + 2\,z + \lambda\,y\,z = 0 \quad . \quad . \quad . \quad . \quad (a)$$

$$x + 2\,z + \lambda\,x\,z = 0 \quad . \quad . \quad . \quad . \quad (b)$$

$$2\,x + 2\,y + \lambda\,x\,y = 0. \quad . \quad . \quad . \quad . \quad (c)$$

Multiply (a) by x, (b) by y and (c) by z, and add; therefore

$$2xy + 4xz + 4yz + 3\lambda xyz = 0;$$

that is,

$$2A + 3\lambda u = 0.$$

$$\therefore \quad \lambda = -\frac{2A}{3u}.$$

Substituting for λ in (a), (b) and (c), we get

$$3xy + 6xz = 2A, \quad 3xy + 6yz = 2A, \quad 6xz + 6yz = 2A.$$

Hence

$$x = y = 2z.$$

If we substitute for y and z in terms of x in the equation

$$xy + 2xz + 2yz = 300,$$

we get

$$x = 10 = y, \quad \text{and} \quad z = 5.$$

Therefore the cistern of maximum capacity will be 10 feet long, 10 feet broad and 5 feet deep.

(108.) **To find the triangle of minimum are which can be described about a circle of radius r.**

The area of the triangle is $= \frac{r}{2}(a + b + c)$, and since is constant the area will be a minimum when $a + b + c$ minimum. It is easy to show that

$$a = r(\cot\tfrac{1}{2}B + \cot\tfrac{1}{2}C)$$
$$b = r(\cot\tfrac{1}{2}A + \cot\tfrac{1}{2}C)$$
$$c = r(\cot\tfrac{1}{2}A + \cot\tfrac{1}{2}B);$$

therefore the triangle will be a minimum when

$$2\cot\tfrac{1}{2}A + 2\cot\tfrac{1}{2}B + 2\cot\tfrac{1}{2}C$$

is a minimum.

We have also

$$A + B + C = 180°,$$

therefore by differentiating we get

$$\operatorname{cosec}^2 \tfrac{1}{2} A \, dA + \operatorname{cosec}^2 \tfrac{1}{2} B \, dB + \operatorname{cosec}^2 \tfrac{1}{2} C \, dC = 0$$

for a minimum, also

$$dA + dB + dC = 0$$

$$\begin{aligned} \therefore \quad \operatorname{cosec}^2 \tfrac{1}{2} A + \lambda &= 0 \\ \operatorname{cosec}^2 \tfrac{1}{2} B + \lambda &= 0 \\ \operatorname{cosec}^2 \tfrac{1}{2} C + \lambda &= 0. \end{aligned}$$

This leads to $A = B = C$, therefore the minimum triangle is equilateral.

EXAMPLES.

1. Three cubic feet of lead are to be formed into the lining of a rectangular cistern, the thickness of the lining is to be $\frac{1}{8}$ inch. Find the dimensions of the cistern so that it may have a maximum capacity, there being no lid.

Ans. Length = breadth = 9·79 feet. Depth = 4·89 feet.

2. Find a point within a triangle, such that the sum of the squares of its distances from the three angles is the least possible.

Ans. The c. g. of the triangle.

3. Find the minimum value of $x^3 y^2 z$ subject to the con-tion $\frac{4}{x} + \frac{5}{y} + \frac{6}{z} = 1$.

$$\textit{Ans.} \quad \text{When } \frac{3x}{4} = \frac{2y}{5} = \frac{z}{6} = 6,$$

$$\text{therefore } x^3 y^2 z = 4147200.$$

4. Given the sum of the three edges of a rectangular ock, find its dimensions such that its surface may be a aximum.

Ans. A cube.

5. Find the maximum value of v when $v = x^2 y^3 z^4$, and $2x + 3y + 4z = c$.

$$\textit{Ans.} \quad \left(\frac{c}{9}\right)^9.$$

6. Divide a line a feet long into three parts, x, y and z, such that the sum of one-half the rectangle xy, one-third the rectangle xz, and one-fourth the rectangle yz shall be a maximum.

$$\textit{Ans.}\quad x = \frac{21a}{47},\ y = \frac{20a}{47} \text{ and } z = \frac{6a}{47}.$$

CHAPTER VIII.

TANGENTS AND NORMALS TO PLANE CURVES.

ART. 28.—Let P and Q be two points on a plane curve A B, and suppose the point Q to move up indefinitely near to P, the straight line drawn through P and Q, when indefinitely near, is called a *tangent* to the curve at the point P.

Let (x', y') and (x'', y'') be the co-ordinates of P and Q respectively; then the equation of the straight line passing through P and Q is

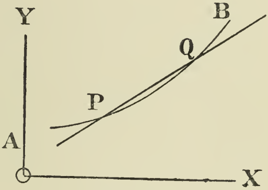


FIG. 9.

$$y - y' = \frac{y'' - y'}{x'' - x'}(x - x').$$

When P and Q are indefinitely near $y'' - y' = d\,y$ and $x'' - x' = d\,x$, therefore the equation of the tangent to a curve at a point (x', y') is

$$y - y' = \frac{d\,y}{d\,x}(x - x'). \quad . \quad . \quad . \quad (a)$$

(109.) **To find the equation of the tangent to the curve whose equation is $x^2 + y^2 = r^2$, at the point (x', y').**

From the equation to the curve we have

$$\frac{d\,y}{d\,x} = -\frac{x}{y} = -\frac{x'}{y'},$$

at the point (x', y').

Substituting in (α) for $\frac{d\,y}{d\,x}$ we get

$$y - y' = -\frac{x'}{y'}(x - x'),$$

that is,

$$y\,y' + x\,x' = x'^2 + y'^2 = r^2.$$

Therefore the equation of the tangent to the curve $x^2 + y^2 = r^2$ at the point (x', y') is

$$x\,x' + y\,y' = r^2.$$

ART. 29.—The *normal* to a curve at the point (x', y') is the straight line drawn through the point perpendicular to the tangent at that point. Its equation is therefore

$$y - y' = -\frac{d\,x}{d\,y}(x - x'). \quad . \quad . \quad . \quad (\beta)$$

(110.) **To find the equation of the normal to the curve** $x^2 + y^2 = r^2$ at the point (x', y').

From the last example we have

$$\frac{d\,y}{d\,x} = -\frac{x'}{y'} \text{ at the point } (x', y'),$$

and on substituting in (β) we have

$$y - y' = \frac{y'}{x'}(x - x'),$$

which reduces to $x\,y' - x'\,y = 0$.

(111.) **To find the equations of the tangent and normal to the curve whose equation is**

$$a\,x^2 + 2\,h\,x\,y + b\,y^2 = k,$$

at the point (x', y').

By Art. 24,

$$\frac{d\,y}{d\,x} = -\frac{\frac{d\,u}{d\,x}}{\frac{d\,u}{d\,y}}.$$

Substituting for $\frac{dy}{dx}$ in (α) we have, for the equation of the tangent,

$$(y - y')\frac{du}{dy} + (x - x')\frac{du}{dx} = 0. \quad . \quad . \quad (\delta)$$

Now,

$$\frac{du}{dy} = 2(hx' + by'), \text{ and } \frac{du}{dx} = 2(ax' + hy');$$

therefore the equation of the tangent at the point (x', y') is

$$(ax' + hy')x + (hx' + by')y = k.$$

The equation of the normal at the point (x', y') is

$$(y - y')\frac{du}{dx} - (x - x')\frac{du}{dy} = 0,$$

that is,

$$(hx' + by')x - (ax' + hy')y = h(x'^2 - y'^2) - (a - b)x'y'.$$

(112.) **To find the tangent and normal to the curve $x^n y^m = c$, at the point (x', y').**

Here we have $n \log x + m \log y = \log c$; therefore

$$\frac{n\,dx}{x} + \frac{m\,dy}{y} = 0;$$

that is,

$$\frac{dy}{dx} = -\frac{ny'}{mx'} \text{ at the point } (x', y').$$

On substituting in the formula (α) we get

$$\frac{nx}{x'} + \frac{my}{y'} = m + n.$$

The normal is given by formula (β), which leads to

$$\frac{mx}{y'} - \frac{ny}{x'} = \frac{mx'}{y'} - \frac{ny'}{x'}.$$

ART. 30.—Subtangents and Subnormals.

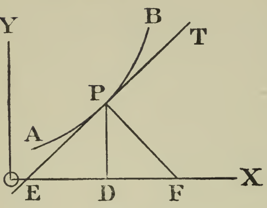


FIG. 10.

Let E P T represent the tangent to the curve A B, at the point P, P F the normal, P D the perpendicular on the axis of X.

Then E D is called the *subtangent*, and D F the *subnormal.*

Denoting the angle P E D by $\theta =$ D P F, we have E D the subtangent $=$ D P $\cot \theta$.

$$\therefore \text{ the subtangent} = y \frac{d\,x}{d\,y} \quad . \quad . \quad (\alpha)$$

and

$$\text{D F the subnormal} = y \frac{d\,y}{d\,x}. \quad . \quad . \quad (\beta)$$

(113.) **Find the subtangent and subnormal to the curve** $y^2 = 4\,a\,x$, at the point (x', y').

Here $\dfrac{d\,y}{d\,x} = \dfrac{2\,a}{y} = \dfrac{y'}{2\,x'}$ at the point (x', y').

Substituting in (α) we get

$$\textit{Subtangent} = y' \times \frac{2\,x'}{y'} = 2\,x',$$

that is, the subtangent for this curve, which is a parabola, is always double the abscissa.

$$\text{The } \textit{subnormal} = y' \times \frac{y'}{2\,x'} = \frac{y'^2}{2\,x'} = 2\,a,$$

that is, the subnormal is constant and equal to twice th distance of the focus from the vertex.

(114.) **Find the subtangent and subnormal to th curve whose equation is** $y = e^{a\,x}$.

Here $\frac{dy}{dx} = a\,e^{ax}$. Therefore

$$Subtangent = e^{ax} \times \frac{1}{a\,e^{ax}} = \frac{1}{a},$$

nd the

$$Subnormal = e^{ax} \times a\,e^{ax} = a\,e^{2ax}.$$

(115.) **Find the subtangent and subnormal to the atenary whose equation is**

$$y = \frac{a}{2}\left\{e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right\}.$$

Here

$$\frac{dy}{dx} = \frac{1}{2}\left\{e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right\}.$$

Therefore the

$$Subtangent = a\,\frac{e^{\frac{x}{a}} + e^{-\frac{x}{a}}}{e^{\frac{x}{a}} - e^{-\frac{x}{a}}},$$

and

$$Subnormal = \frac{a}{4}\left\{e^{\frac{2x}{a}} - e^{-\frac{2x}{a}}\right\}.$$

ART. 31.—In Fig. 10 of the last article, P E is called the *length of the tangent* and P F the *length of the normal.*

Now,

$$\text{P E} = y \operatorname{cosec} \theta = y\sqrt{1 + \left(\frac{dx}{dy}\right)^2}.$$

$$\therefore\ \textbf{Tangent} = y\sqrt{\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}}.$$

And

$$\text{P F, the } \textbf{Normal} = y\sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}.$$

(116). **Find the tangent and normal to the catenary,**

$$y = \frac{a}{2}\left\{e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right\}.$$

Here

$$\frac{dy}{dx} = \frac{1}{2}\left\{e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right\},$$

therefore

$$\textbf{Tangent} = \frac{a}{2}\,\frac{\left\{e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right\}^2}{\left\{e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right\}}.$$

$$\textbf{Normal} = \frac{a}{2}\left\{e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right\}\sqrt{1 + \frac{1}{4}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right)}$$

$$= \frac{a}{4}\left\{e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right\}^2$$

$$= \frac{y^2}{a}.$$

EXAMPLES.

1. Find the equation of the tangent to the parabola $y^2 = 4ax$, at the end of the latus rectum ($x = a$).

Ans. $x - y + a = 0$.

2. Find the equation of the normal to the parabola $y^2 = 4ax$, at the end of the latus rectum.

Ans. $x + y - 3a = 0$.

3. Find the length of the subtangent and subnormal to the parabola $y^2 = 4ax$, at the point ($x = a$).

Ans. Subtangent $= 2a$.
Subnormal $= 2a =$ constant for all points on the curve.

4. Find the equation of the tangent to the curve $y = a^x$ at the point $x = x^1$.

Ans. $y = a^{x'}\left\{(x - x^1)\log a + 1\right\}$.

5. Find the equation of the tangent to the ellipse $+ \frac{y^2}{b^2} = 1$, at the point $\left\{ \begin{matrix} x = a\,e \\ y = b\sqrt{1 - e^2} \end{matrix} \right\}$.

Ans. $b\,e\,x + a\sqrt{(1 - e^2)}\,y - a\,b = 0.$

6. Find the length of the subtangent and subnormal to he ellipse mentioned in Example 5, at the point $x = a\,e$, $= b\sqrt{1 - e^2}$.

$$\textit{Ans.} \quad \textit{Subtangent} = \frac{a}{e}(1 - e^2).$$

$$\textit{Subnormal} = \frac{e\,b^2}{a}.$$

7. Find the equation of the tangent to the curve $x^a\,y^b = c$, t the point $x = 1$, $y = \sqrt[b]{c}$.

$$\textit{Ans.} \quad y + \frac{a}{b}\sqrt[b]{c}\,x = \sqrt[b]{c}\left(1 + \frac{a}{b}\right).$$

8. Find the equation of the tangent to the curve $^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}$, at the point $x = a$.

Ans. $(c^{\frac{2}{3}} - a^{\frac{2}{3}})^{\frac{1}{2}}\,x + a^{\frac{1}{3}}\,y = (c^{\frac{2}{3}} - a^{\frac{2}{3}})^{\frac{1}{2}}\,c^{\frac{2}{3}}\,a^{\frac{1}{3}}.$

CHAPTER IX.

RADIUS OF CURVATURE OF PLANE CURVE

ART. 32.—Let A B (Fig. 11) be part of a plane curve, ar let P O′, Q O′ be two normals at the points P and Q inte secting at O′; then if Q a proaches indefinitely near t P, O′ P and O′ Q will be equ to one another, and the circl described with O′ as centre an radius O′ P is called the *circl* *of curvature*, O′ P is called the *radius of curvature*, and O′ the centre of curvature of the curve A B at the point P.

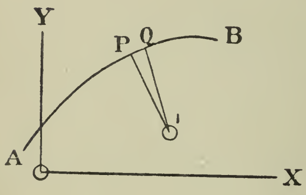


FIG. 11.

Denoting the arc P Q by ds, the angle P O′ Q by $d\theta$, and O′ P by r, then

$$ds = r\,d\theta,$$

or

$$\frac{1}{r} = \frac{d\theta}{ds}. \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (a)$$

Now, θ is the angle which the tangent, at P, makes with the axis of X, and $d\theta$ is equal to the angle between the tangents at P and Q. We have also

$$\frac{dy}{dx} = \tan\theta,$$

hence

$$\frac{d^2y}{dx^2} = \sec^2\theta\,\frac{d\theta}{dx},$$

since θ is a function of x; therefore

$$\frac{d\,\theta}{d\,x} = \cos^2\theta\,\frac{d^2 y}{d\,x^2}.$$

Also

$$\frac{d\,x}{d\,s} = \cos\vartheta.$$

Again,

$$\frac{1}{r} = \frac{d\,\theta}{d\,s} = \frac{d\,\theta}{d\,x} \times \frac{d\,x}{d\,s} = \cos^3\theta\,\frac{d^2 y}{d\,x^2}$$

$$= \frac{\frac{d^2 y}{d\,x^2}}{\sec^3\theta} = \frac{\frac{d^2 y}{d\,x^2}}{\{1 + \tan^2\theta} = \frac{\frac{d^2 y}{d\,x^2}}{\left\{1 + \left(\frac{d\,y}{d\,x}\right)^2\right\}^{\frac{3}{2}}}.$$

$$\therefore\ r = \frac{\left\{1 + \left(\frac{d\,y}{d\,x}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2 y}{d\,x^2}}. \quad . \quad . \quad . \quad (\beta)$$

Since the numerator of this expression may be positive negative, it is necessary to give to it the same sign as $\frac{y}{{}^2}$ in order that r may be positive.

If the curve be convex or concave to the axis of X, $\frac{d^2 y}{d\,x^2}$ be positive or negative respectively.

If $\frac{d^2 y}{d\,x^2} = 0$, the radius of curvature will be infinite. At oint of inflexion where the curve changes from convex to ave to the axis of X, or *vice versâ*, $\frac{d^2 y}{d\,x^2} = 0$.

t is evident that $\frac{d\,y}{d\,x}$ is the same for both the curve and ircle at the point P, and $\frac{d^2 y}{d\,x^2}$ is also the same for both, since the circle and the curve A B have a common normal at the points P and Q.

The circle of curvature is said to have contact of the second order, since $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ are the same for both.

If two curves given by the equations $y = f(x)$ and $y = \phi(x)$ touch each other at the point $x = a$, then $f^{\mathrm{I}}(a) = \phi^{\mathrm{I}}(a)$; and if $f^{\mathrm{II}}(a) = \phi^{\mathrm{II}}(a)$, and also $f^{\mathrm{III}}(a) = \phi^{\mathrm{III}}(a)$, they are said to have contact of the third order; and if $f^n(a) = \phi^n(a)$, they are said to have contact of the nth order.

The reciprocal of the radius of curvature is called *curvature.*

(117.) **To find the radius of curvature of the ellipse whose equation is**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

at the end of the major axis.

Here

$$\frac{dy}{dx} = -\frac{b^2}{a^2}\frac{x}{y},$$

and

$$\frac{d^2y}{dx^2} = -\frac{b^2}{a^2}\frac{y - x\dfrac{dy}{dx}}{y^2}.$$

Substituting these values in (β), we get

$$r = \frac{(b^4x^2 + a^4y^2)^{\frac{3}{2}}}{(ab)^4}.$$

At the extremity of the major axis

$$x = a \quad \text{and} \quad y = 0.$$

$$\therefore \; r = \frac{b^2}{a}.$$

At the extremity of the minor axis

$$x = 0 \quad \text{and} \quad y = b.$$

$$\therefore \quad r = \frac{a^2}{b}.$$

(118.) **To find the radius of curvature at the vertex of the parabola** $y^2 = 4\,a\,x$.

Here

$$\frac{d\,y}{d\,x} = \frac{2\,a}{y},$$

and

$$\frac{d^2\,y}{d\,x^2} = -\frac{2\,a\,\frac{d\,y}{d\,x}}{y^2} = -\frac{4\,a^2}{y^3}.$$

Substituting in (β), we have

$$r = \frac{(y^2 + 4\,a^2)^{\frac{3}{2}}}{4\,a^2}.$$

At the vertex $y = 0$.

$$\therefore \quad r = 2\,a.$$

(119.) **To find the radii of curvature of the curve**

$$y = (x - 1)\,(x - 2)\,(x - 3)$$

at the points where

$$x = 1, \quad x = 2 \quad \text{and} \quad x = 3.$$

Here

$$\frac{d\,y}{d\,x} = 3\,x^2 - 12\,x + 11,$$

and

$$\frac{d^2\,y}{d\,x^2} = 6\,(x - 2).$$

Where $x = 1$, $\frac{d\,y}{d\,x} = 2$ and $\frac{d^2 y}{d\,x^2} = -6$.

$$\therefore \quad r = \frac{\left\{1 + \left(\frac{d\,y}{d\,x}\right)^2\right\}}{\frac{d^2 y}{d\,x^2}} = \frac{(5)^{\frac{3}{2}}}{6},$$

taking the negative root of the numerator.

Where $x = 2$, $\frac{d^2 y}{d\,x^2} = 0$.

$$\therefore \quad r = \infty,$$

a point of inflexion.

Where $x = 3$, $\frac{d\,y}{d\,x} = 2$ and $\frac{d^2 y}{d\,x^2} = 6$,

$$\therefore \quad r = \frac{(5)^{\frac{3}{2}}}{6}.$$

The radii have the same numerical value at the points $x = 1$ and $x = 3$, but the centres of curvature for the tw points are on opposite sides of the curve, since $\frac{d^2 y}{d\,x^2}$ is opposit in sign for the two points.

(120.) **To find the radius of curvature of the curv** $y = e^x$, where $x = 0$.

Here $\frac{d\,y}{d\,x} = e^x$, and $\frac{d^2 y}{d\,x^2} = e^x$, where $x = 0$, $\frac{d\,y}{d\,x} = 1$, and $\frac{d^2 y}{d\,x^2} = 1$.

$$\therefore \quad r = 2^{\frac{3}{2}} = 2\sqrt{2}.$$

(121.) **To find the radius of curvature of the catenary,**

$$y = \frac{a}{2}\left\{e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right\}$$

at the vertex, $x = 0$.

Here

$$\frac{d\,y}{d\,x} = \frac{1}{2}\left\{e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right\},$$

and

$$\frac{d^2\,y}{d\,x^2} = \frac{1}{2\,a}\left\{e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right\} = \frac{y}{a^2}.$$

$$\therefore\quad r = \frac{\left\{1 + \left(\frac{d\,y}{d\,x}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2\,y}{d\,x^2}} = \frac{\frac{y^3}{a^3}}{\frac{y}{a^2}} = \frac{y^2}{a},$$

where $x = 0$, $y = a$; therefore the radius of curvature at the vertex is equal to a.

(122.) **To find the radius of curvature of the curve $y = x^4 - 8\,x^3 + x^2$, at the origin.**

Here $\frac{d\,y}{d\,x} = 4\,x^3 - 24\,x^2 + 2\,x$, and $\frac{d^2\,y}{d\,x^2} = 12\,x^2 - 48\,x + 2$. At the origin $x = 0$.

$$\therefore\quad \frac{d\,y}{d\,x} = 0, \quad \text{and} \quad \frac{d^2\,y}{d\,x^2} = 2.$$

$$\therefore\quad r = \frac{1}{2}.$$

(123.) **Find the point of inflexion in the curve $y = x^3 - 15\,x^2 + 36\,x + 7$.**

We have to find that value of x which will render $\frac{d^2\,y}{d\,x^2} = 0$.

Now,

$$\frac{d\,y}{d\,x} = 3\,x^2 - 30\,x + 36,$$

and

$$\frac{d^2\,y}{d\,x^2} = 6\,(x - 5).$$

$$\therefore\quad x = 5$$

is the point of inflexion.

Art. 33.—**To find the centre of curvature of a curve at any point on it.**

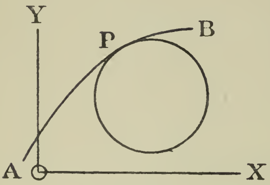


Fig. 12.

Let the co-ordinates of the centre of the circle, touching the curve A B at P, be (h, k).

Then its equation is

$$(x - h)^2 + (y - k)^2 = r^2;$$

therefore

$$x - h + (y - k)\frac{dy}{dx} = 0,$$

and

$$1 + \left(\frac{dy}{dx}\right)^2 + (y - k)\frac{d^2 y}{dx^2} = 0;$$

therefore

$$k = y + \frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2 y}{dx^2}},$$

the ordinate of the centre; and

$$h = x - \frac{\frac{dy}{dx}\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}{\frac{d^2 y}{dx^2}},$$

the abscissa of the centre.

(124.) **To find the co-ordinates of the centre of curvature of the curve $y = e^{ax}$.**

Here

$$\frac{dy}{dx} = a\,e^{ax} \quad \text{and} \quad \frac{d^2 y}{dx^2} = a^2\,e^{ax}.$$

$$\therefore \quad h = \frac{a x - a^2 e^{2ax} - 1}{a},$$

and

$$k = \frac{2\,a^2\,e^{2\,a\,x} + 1}{a^2\,e^{a\,x}}.$$

Where $x = 0$,

$$h = -\left(a + \frac{1}{a}\right),$$

and

$$k = \frac{2\,a^2 + 1}{a^2}.$$

EXAMPLES.

1. Show that the radius of curvature of a circle is constant, using the general formula.

2. Find the radius of curvature of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ at the point $(x'\,y')$.

Ans. $r = 3\,(a\,x'\,y')^{\frac{1}{3}}$.

3. Find the radius of curvature of the curve whose equation, $y = a \sin x$, at the point $x = \frac{\pi}{2}$.

Ans. $r = \frac{1}{a}$.

4. Find the radius of curvature of the curve whose equation is $y = e^{\tan x}$ at the point $x = \frac{\pi}{4}$.

Ans. $r = \frac{(1 + 4\,e^2)^{\frac{3}{4}}}{8\,e}$.

5. Find the radius of curvature of the curve whose equation is $y = x^5 - x^4 - 3\,x - 1$ at the point $x = 1$.

Ans. $r = \frac{\sqrt{125}}{8}$.

6. Find the radius of curvature of the curve whose equation is

$$y = a \cos^{-1}\left(\frac{a - x}{a}\right) + \sqrt{(2\,a\,x - x^2)}$$

at the point $x = a$.

Ans. $r = 2\sqrt{2}\,a$.

7. Find the co-ordinates of the centre of the curve $y^2 = 4\,a\,x$.

Ans. $h = 2\,a + 3\,x$.

$$k = -\frac{2\,x^{\frac{3}{2}}}{\sqrt{a}}.$$

8. Find the curvature at the point $x = 0$ in the curve $x = y^4 - 4\,y^3 - 18\,y^2$.

Ans. $r = \frac{1}{36}$.

9. Find the radius of curvature of $x\,y = 4$.

Ans. $r = \frac{(x^2 + y^2)^{\frac{3}{2}}}{8}$.

10. Find the radius of curvature at the point 0, 0 in the curve $y^4 + y^2 - y + x^3 + x^2 = 0$.

Ans. $r = \frac{1}{2}$.

11. Prove that the curvature at the origin in the curve $3\,y = 2\,x^3 + 4\,x^2\,y$ is 0.

CHAPTER X.

RT. 34.—The Cycloid, Epicycloid and Hypocycloid.

The cycloid *is the path described by a point on the ircumference of a circle as it rolls along a straight line.*

Let the circle P D, Fig. 13, roll along the line O X. The oint P, which was initially at O, will describe a cycloidal urve O P Q.

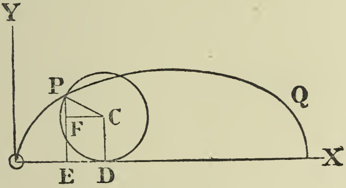


FIG. 13.

Let O E $= x$, P E $= y$, and the angle P C D $= \theta$. Drop C F perpendicular to P E, which is perpendicular to O X. Now, $y = \mathrm{EF} + \mathrm{FP} = a + a \cos \mathrm{FPC} = a\,(1 - \cos\theta)$, where a is the radius of the generating circle. Again, $x = \mathrm{OD} - \mathrm{ED} = a\,\theta - a \sin\theta = a\,(\theta - \sin\theta)$; therefore the cycloid is given by the equations

$$\left.\begin{aligned} x &= a\,(\theta - \sin\theta) \\ y &= a\,(1 - \cos\theta) \end{aligned}\right\}.$$

If we eliminate θ we have

$$x = a \operatorname{vers}^{-1}\left(\frac{y}{a}\right) - \sqrt{(2\,a\,y - y^2)}.$$

he direction of the tangent to the cycloid at any point P, $\frac{d\,y}{d\,x}$.

Now,

$$d x = a\,(1 - \cos\theta)\,d\theta,$$

and

$$d y = a \sin\theta\, d\theta;$$

therefore

$$\frac{d y}{d x} = \frac{\sin\theta}{1 - \cos\theta} = \cot \tfrac{1}{2}\,\theta.$$

Art. 35.—**The Epicycloid** *is the path described by point on the circumference of a circle as it rolls externally roun the circumference of another circle.*

Let A B C be a fixed circle whose centre is taken as origi of co-ordinates, and let its radius be r. Let P B G be th rolling circle, and let P be th point on the circumference o the circle P B G which trace out the epicycloid.

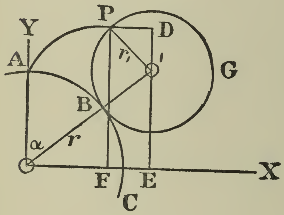


Fig. 14.

The point P, at starting, coincides with A; therefore the arc A B = arc P B, and if the angle A O B = α, and the angle P O′ B = β, therefore $r\,\alpha = r_1\,\beta$, where r_1 is the radius of the circle P B G.

Let x and y be the co-ordinates of the point P; then

$$x = \mathrm{O\,F} = \mathrm{O\,E} - \mathrm{P\,D}.$$

$$\therefore\quad x = (r + r_1)\sin\alpha - r_1 \sin(\alpha + \beta),$$

and

$$y = \mathrm{P\,F} = \mathrm{O'\,E} + \mathrm{O'\,D}.$$

$$\therefore\quad y = (r + r_1)\cos\alpha - r_1 \cos(\alpha + \beta).$$

If we substitute $\dfrac{r\,\alpha}{r_1}$ for β, we get

$$\left.\begin{aligned} x &= (r + r_1)\sin\alpha - r_1 \sin\left(\frac{r + r_1}{r_1}\right)\alpha. \\ y &= (r + r_1)\cos\alpha - r_1 \cos\left(\frac{r + r_1}{r_1}\right)\alpha. \end{aligned}\right\}$$

If we differentiate, we have

$$\frac{d\,y}{d\,a} = (r + r_1)\left\{\sin\left(\frac{r + r_1}{r_1}\right) a - \sin a\right\}$$

$$\frac{d\,x}{d\,a} = (r + r_1)\left\{\cos a - \cos\left(\frac{r + r_1}{r_1}\right) a\right\}.$$

$$\therefore\quad \frac{d\,y}{d\,x} = \cot\left(\frac{r + 2\,r_1}{2\,r_1}\right) a.$$

This gives the direction of the tangent to the curve when a is given.

ART. 36.—**The hypocycloid** *is the path described by a point on the circumference of a circle as it rolls internally round the circumference of another circle.*

Let A B C be the fixed circle, whose centre is taken as origin of co-ordinates, and let its radius be r, P B G the rolling circle whose radius is r_1, P the tracing point which coincides with A at starting; therefore the arc B P = arc A B.

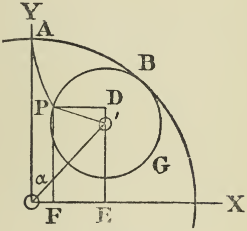


FIG. 15.

Let a denote the angle A O O′, and β the angle π − P O′ O.

$$x = \text{O F} = \text{O E} - \text{P D}.$$

$$\therefore\quad x = (r - r_1)\sin a - r_1 \sin(a + \pi - \beta),$$

and

$$y = \text{P F} = \text{O}'\text{E} + \text{O}'\text{D}.$$

$$\therefore\quad y = (r - r_1)\cos a - r_1 \cos(a + \pi - \beta).$$

But $r\,a = r_1\,\beta$; therefore, substituting $\frac{r\,a}{r_1}$ for β, we have

$$\left.\begin{aligned} x &= (r - r_1)\sin a - r_1 \sin\left(\frac{r - r_1}{r_1}\right) a. \\ y &= (r - r_1)\cos a + r_1 \cos\left(\frac{r - r_1}{r_1}\right) a. \end{aligned}\right\}$$

$$\frac{dy}{da} = -(r - r_1)\left\{\sin a + \sin\left(\frac{r - r_1}{r_1}\right)a\right\}.$$

$$\frac{dx}{da} = (r - r_1)\left\{\cos a - \cos\left(\frac{r - r_1}{r_1}\right)a\right\}.$$

$$\therefore\quad \frac{dy}{dx} = -\cot\left(\frac{r - 2r_1}{2r_1}\right)a.$$

CHAPTER XI.

THE INTEGRAL CALCULUS.

Integration.

ART. 37.—The process of integrating may be said to be the finding of a function when its differential is given.

It may also be considered as the finding of the sum of an infinite series whose terms are indefinitely small.

The student, who has already mastered the elements of the Differential Calculus, will have no trouble in writing down the integrals of a number of elementary differentials.

The symbol used to denote the process of integration is $\int$ placed immediately before the expression to be integrated. For example: $y = \int x^n\, d\, x + \mathrm{C}$ means that y is the integral of the differential $x^n\, d\, x$, where C is an arbitrary constant; since the differentials of $\dfrac{x^{n+1}}{n+1}$ and $\dfrac{x^{n+1}}{n+1} + \mathrm{C}$ are both $x\ d\, x$; therefore, in order to give the general form of an integral, it is necessary to add a constant, except in the case of definite integrals, which will be explained subsequently. We may here give a list of elementary integrals, the constant in each case being understood.

$$\int x^n\, d\, x = \frac{x^{n+1}}{n+1}. \quad . \quad . \quad . \quad . \qquad (A)$$

(That is, increase the index by unity and divide by the index thus increased.)

$$\int \frac{d\,x}{x} = \log x. \quad \ldots \quad (B)$$

$$\int e^{a\,x}\,d\,x = \frac{1}{a}\,e^{a\,x}. \quad \ldots \quad (C)$$

$$\int a^x\,d\,x = a^x \log_a e. \quad \ldots \quad (D)$$

$$\int \sin a\,x\,d\,x = -\frac{1}{a}\cos a\,x. \quad \ldots \quad (E$$

$$\int \cos a\,x\,d\,x = \frac{1}{a}\sin a\,x. \quad \ldots \quad (F)$$

$$\int \sec^2 a\,x\,d\,x = \frac{1}{a}\tan a\,x. \quad \ldots \quad (G)$$

$$\int \operatorname{cosec}^2 a\,x\,d\,x = -\frac{1}{a}\cot a\,x. \quad \ldots \quad (H)$$

$$\int \frac{d\,x}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} = -\cos^{-1}\frac{x}{a}. \quad . \quad (I)^*$$

$$\int \frac{d\,x}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}. \quad \ldots \quad (J)$$

$$\int \frac{d\,x}{\sqrt{a^2 + x^2}} = \log\left(x + \sqrt{a^2 + x^2}\right). \quad . \quad (K)$$

$$\int \frac{f'(x)\,d\,x}{f(x)} = \log f(x). \quad \ldots \quad (L)$$

These integrals, with the exception of the last two, are called *elementary integrals*, since our knowledge of the Differential Calculus at once suggests them. The integrals K and L are added for the sake of reference. We shall give the working out of K and L.

(125.) Let $\sqrt{a^2 + x^2} = v - x$. By squaring, we get $a^2 = v^2 - 2\,v\,x$, and, taking the differentials, we get

$$2\,v\,d\,v - 2\,v\,d\,x - 2\,x\,d\,v = 0.$$

---

* See Differential Calculus, Art. 10.

$$\therefore \quad dx = \left(\frac{v-x}{v}\right) dv.$$

Substituting for $d x$ and $\sqrt{(a^2 + x^2)}$ in $\int \frac{d x}{\sqrt{a^2 + x^2}}$, it becomes

$$\int \left(\frac{v-x}{v}\right) \frac{d v}{(v-x)} = \int \frac{d v}{v} = \log v = \log (x + \sqrt{a^2 + x^2})$$

by B.

(126.) Let $f(x) = z$; therefore

$$d z = f'(x)\, d x.$$

$$\therefore \quad \int \frac{f'(x)\, d x}{f(x)} \text{ becomes } \int \frac{d z}{z} = \log z = \log f(x) \text{ by B.}$$

ART. 38.—Integration by Substitution.

In Examples (125) and (126) we employed the method substitution in order to reduce the differentials to the re elementary forms. This method can be made use of reduce many differentials to one or other of the elementary ms. We shall add a few examples on its application.

(127.) **To integrate** $(a + x)^{\frac{1}{3}}\, d x$.

Let $a + x = v$; therefore $d x = d v$.

$$\therefore \quad \int (a + x)^{\frac{1}{3}}\, d x \text{ becomes } \int v^{\frac{1}{3}}\, d v = \frac{3}{4} v^{\frac{4}{3}} = \frac{3}{4} (a + x)^{\frac{4}{3}}$$

y A.

(128.) **To integrate** $\frac{x^n\, d x}{(a x + b)^m}$ **where** n **is a positive teger.**

Let $a x + b = v$; therefore $x = \frac{v - b}{a}$ and $d x = \frac{d v}{a}$.

stituting for $a x + b$ and $d x$ in terms of v,

$$\int \frac{x^n\, d x}{(a x + b)^m} \text{ becomes } \frac{1}{a^{n+1}} \int \frac{(v - b)^n}{v^m}\, d v.$$

Expanding by the Binomial Theorem we get

$$\frac{1}{a^{n+1}}\int\left(v^{n-m} - n\,b\,v^{n-m-1} + \frac{n\,(n-1)}{\underline{|2}}\,b^2\,v^{n-m-2} - \&c.\right)d\,v$$

$$= \frac{1}{a^{n+1}}\left\{\int v^{n-m}\,d\,v - n\,b\int v^{n-m-1}\,d\,v\right.$$

$$\left. + \frac{n\,(n-1)}{\underline{|2}}\,b^2\int v^{n-m-2}\,d\,v - \&c.\right\}.$$

$$= \frac{1}{a^{n+1}} \times \frac{v^{n-m+1}}{n-m+1} - \frac{n\,b}{a^{n+1}} \times \frac{v^{n-m}}{n-m} + \&c.$$

In this example, if $n = 3$, $m = 2$, $a = 2$, and $b = 1$,

$$\int \frac{x^3\,d\,x}{(2\,x+1)^2} \quad \text{becomes} \quad \frac{1}{16}\int\frac{(v-1)^3}{v^2}\,d\,v$$

$$= \frac{1}{16}\int\left(v - 3 + \frac{3}{v} - \frac{1}{v^2}\right)d\,v$$

$$= \frac{1}{16}\left\{\int v\,d\,v - 3\int d\,v + 3\int\frac{d\,v}{v} - \int\frac{d\,v}{v^2}\right\}$$

$$= \frac{1}{16}\left\{\frac{v^2}{2} - 3\,v + 3\log v + \frac{1}{v}\right\}$$

$$= \frac{(2\,x+1)^2}{32} - \frac{3}{16}(2\,x+1) + \frac{3}{16}\log\,(2\,x+1) + \frac{(2\,x+1)^{-}}{16}$$

(129.) **To integrate** $\dfrac{d\,x}{\sqrt{a\,x^2 + 2\,b\,x + c}}$.

Here

$$\int\frac{d\,x}{\sqrt{a\,x^2+2\,b\,x+c}} = \sqrt{a}\int\frac{d\,x}{\sqrt{(a^2\,x^2 + 2\,a\,b\,x + a\,c)}}$$

$$= \sqrt{a}\int\frac{d\,x}{\sqrt{(a\,x+b)^2 + a\,c - b^2}}.$$

Let $a\,x + b = v$; therefore $d\,x = \dfrac{d\,v}{a}$. Substituting or

dx and $ax + b$ in terms of v, we get

$$\frac{1}{\sqrt{a}}\int\frac{dv}{\sqrt{v^2 + ac - b^2}} = \frac{1}{\sqrt{a}}\log(v + \sqrt{v^2 + ac - b^2})$$

$$= \frac{1}{\sqrt{a}}\log\{ax + b + \sqrt{a(ax^2 + 2bx + c)}\}$$

by K, where $ac > b^2$.

(130.) **To integrate** $\dfrac{dx}{ax^2 + 2bx + c}$.

Here

$$\int\frac{dx}{ax^2 + 2bx + c} = a\int\frac{dx}{(ax + b)^2 + ac - b^2}.$$

Assume $v = ax + b$; therefore $\dfrac{dv}{a} = dx$; therefore, by substitution the expression becomes

$$\int\frac{dv}{v^2 + ac - b^2} = \frac{1}{\sqrt{ac - b^2}}\tan^{-1}\frac{v}{\sqrt{ac - b^2}}$$

$$= \frac{1}{\sqrt{ac - b^2}}\tan^{-1}\frac{(ax + b)}{\sqrt{ac - b^2}}$$

by J. This is assuming that $ac - b^2$ is positive. If $ac - b^2$ be negative and equal to $-d^2$ say, then

$$\int\frac{dv}{v^2 + ac - b^2} \quad \text{becomes} \quad \int\frac{dv}{v^2 - d^2}.$$

But

$$\frac{1}{v^2 - d^2} = \frac{1}{2d}\left\{\frac{1}{v - d} - \frac{1}{v + d}\right\}.$$

Therefore

$$\int\frac{dv}{v^2 - d^2} = \frac{1}{2d}\int\frac{dv}{v - d} - \frac{1}{2d}\int\frac{dv}{v + d}$$

$$= \frac{1}{2d}\{\log(v - d) - \log(v + d)\}$$

$$= \frac{1}{2d}\log\frac{v - d}{v + d}$$

$$= \frac{1}{2\sqrt{b^2 - ac}}\log\frac{ax + b - \sqrt{b^2 - ac}}{ax + b + \sqrt{b^2 - ac}}.$$

(131.) **To integrate** $\dfrac{dx}{\sqrt{c - 2bx - ax^2}}$.

$$\int \frac{dx}{\sqrt{c - 2bx - ax^2}} = \sqrt{a} \int \frac{dx}{\sqrt{ac + b^2 - (b + ax)^2}}.$$

Let $v = ax + b$; therefore $\dfrac{dv}{a} = dx$. Substituting, we get

$$\frac{1}{\sqrt{a}} \int \frac{dv}{\sqrt{ac + b^2 - v^2}} = \frac{1}{\sqrt{a}} \sin^{-1} \frac{v}{\sqrt{ac + b^2}}$$

$$= \frac{1}{\sqrt{a}} \sin^{-1} \left(\frac{ax + b}{\sqrt{ac + b^2}} \right),$$

by **I.**

(132.) **To integrate** $\dfrac{dx}{x\sqrt{ax^2 + 2bx + c}}$.

Let $x = \dfrac{1}{z}$; therefore $\dfrac{dx}{x} = -\dfrac{dz}{z}$; and by substituting for x in terms of z,

$$\int \frac{dx}{x\sqrt{ax^2 + 2bx + c}} = -\int \frac{dz}{\sqrt{a + 2bz + cz^2}}$$

$$= -\frac{1}{\sqrt{c}} \log \{cz + b + \sqrt{c(cz^2 + 2bz + a)}\},$$

by 129, if c be positive. If c be negative, it reduces to 131. Substituting $\dfrac{1}{x}$ for z, we get the required integral.

(133.) **To integrate** $\dfrac{(h + kx)\,dx}{ax^2 + 2bx + c}$.

Here

$$\int \frac{(h + kx)\,dx}{ax^2 + 2bx + c} = \int \frac{\left(kx + \dfrac{kb}{a} + h - \dfrac{kb}{a}\right)}{ax^2 + 2bx + c}\,dx$$

$$= \frac{k}{a} \int \frac{(ax + b)\,dx}{ax^2 + 2bx + c} + \frac{ah - kb}{a} \int \frac{dx}{ax^2 + 2bx + c}$$

$$= \frac{k}{2a} \log (a x^2 + 2 b x + c) + \frac{a h - k b}{a}$$

$$\times \frac{1}{\sqrt{a c - b^2}} \tan^{-1}\left(\frac{a x + b}{\sqrt{a c - b^2}}\right),$$

by L and Example 130, if $a c - b^2$ be positive. If $a c - b^2$ be negative, the second expression on the right will be

$$\frac{a h - k b}{a} \times \frac{1}{2 \sqrt{b^2 - a c}} \log \frac{a x + b - \sqrt{b^2 - a c}}{a x + b + \sqrt{b^2 - a c}},$$

by Example 130, second case.

(134.) **To integrate** $\frac{d x}{(a x^2 + b)^{\frac{3}{2}}}$.

Let $v = \frac{1}{x}$; therefore $d x = - \frac{d v}{v^2}$, and $a x^2 + b = \frac{a + b v^2}{v^2}$. Substituting in terms of v, the required integral becomes

$$- \int \frac{v \, d v}{(b v^2 + a)^{\frac{3}{2}}}.$$

Again, assume $b v^2 + a = z^2$; therefore $v \, d v = \frac{z \, d z}{b}$, and we get

$$- \int \frac{v \, d v}{(b v^2 + a)^{\frac{3}{2}}} = - \frac{1}{b} \int \frac{z \, d z}{z^3} = - \frac{1}{b} \int \frac{d z}{z^2}$$

$$= \frac{1}{b z} = \frac{1}{b (b v^2 + a)^{\frac{1}{2}}} = \frac{x}{b (a x^2 + b)^{\frac{1}{2}}}.$$

(135.) **To integrate** $\frac{d x}{\sin x}$.

Here

$$\int \frac{d x}{\sin x} = \int \frac{\sin x \, d x}{1 - \cos^2 x}.$$

Let $v = \cos x$; therefore $dv = -\sin x\, dx$, and on substituting in terms of v we have

$$\int \frac{dx}{\sin x} = -\int \frac{dv}{1 - v^2} = \frac{1}{2} \log \frac{1 - v}{1 + v}$$

$$= \log \sqrt{\frac{1 - \cos x}{1 + \cos x}} = \log \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} = \log \tan \frac{x}{2}.$$

(136.) **To integrate** $\dfrac{dx}{\cos x}$.

Here

$$\int \frac{dx}{\cos x} = \int \frac{\cos x\, dx}{1 - \sin^2 x} = \int \frac{dv}{1 - v^2}$$

where $v = \sin x$, and

$$\int \frac{dv}{1 - v^2} = \frac{1}{2} \log \frac{1 + v}{1 - v} = \log \sqrt{\frac{1 + \sin x}{1 - \sin x}}$$

$$= \log \frac{\sin \frac{x}{2} + \cos \frac{x}{2}}{\cos \frac{x}{2} - \sin \frac{x}{2}} = \log \tan \left(\frac{\pi}{4} + \frac{x}{2}\right).$$

(137.) **To integrate** $\dfrac{dx}{b + c \cos x}$.

$$\int \frac{dx}{b + c \cos x} = \int \frac{dx}{b + c\left(\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}\right)}$$

$$= \int \frac{dx}{(b + c) \cos^2 \frac{x}{2} + (b - c) \sin^2 \frac{x}{2}} = \int \frac{\sec^2 \frac{x}{2}\, dx}{(b + c) + (b - c) \tan^2 \frac{x}{2}}$$

Let $v = \tan\frac{x}{2}$ and the expression becomes

$$2\int\frac{d\,v}{(b+c)+(b-c)\,v^2} = \frac{2}{\sqrt{b^2-c^2}}\tan^{-1}\left\{\sqrt{\frac{b-c}{b+c}}\tan\frac{x}{2}\right\}$$

by 130, if b be greater than c.

If b be less than c, we have

$$2\int\frac{d\,v}{(b+c)+(b-c)v^2} = \frac{1}{\sqrt{c^2-b^2}}\log\frac{\sqrt{c+b}+\sqrt{c-b}\tan\frac{x}{2}}{\sqrt{c+b}-\sqrt{c-b}\tan\frac{x}{2}}$$

by 130, second case.

Examples.

1. $\int x^3\,d\,x$. *Ans.* $\frac{x^4}{4}$.

2. $\int(x^3-3\,x^2+2\,x+1)\,d\,x$.

Ans. $\frac{x^4}{4}-x^3+x^2+x$.

3. $\int\left(2\,x^{\frac{1}{3}}-\frac{1}{x}+\sin x\right)d\,x$. *Ans.* $\frac{3}{2}x^{\frac{4}{3}}-\log x-\cos x$.

4. $\int(a^x+a^{-x})\,d\,x$. *Ans.* $\text{Log}_a\,e\,(a^x-a^{-x})$.

5. $\int\frac{d\,x}{1-x}$. *Ans.* $-\log(1-x)$.

6. $\int(e^{3x}-a^{2x}-3)\,d\,x$.

Ans. $\frac{1}{3}e^{3x}-\frac{1}{2}a^{2x}\log_a e-3\,x$.

7. $\int\frac{x\,d\,x}{1+x^2}$. *Ans.* $\frac{1}{2}\log(1+x^2)$.

8. $\int\frac{x^2\,d\,x}{1+x^3}$. *Ans.* $\frac{1}{3}\log(1+x^3)$.

9. $\int \frac{\cos x \, d x}{1 + \sin x}$. *Ans.* $\text{Log}\,(1 + \sin x)$.

10. $\int \frac{e^{-x} \, d x}{a + b\, e^{-x}}$. *Ans.* $-\frac{1}{b} \log\,(a + b\, e^{-x})$.

11. $\int \frac{x \, d x}{\sqrt{1 + x^2}}$. *Ans.* $\sqrt{1 + x^2}$.

12. $\int \frac{x^{-1} \, d x}{\log x}$. *Ans.* $\text{Log}\,(\log x)$.

13. $\int \sin a x \sin b x \, d x$.

Ans. $\frac{1}{2} \frac{\sin\,(a - b)\, x}{a - b} - \frac{1}{2} \frac{\sin\,(a + b)\, x}{a + b}$.

14. $\int \frac{(1 + x)^3 \, d x}{x^2}$. *Ans.* $\frac{x^2}{2} + 3 x + 3 \log x - \frac{1}{x}$

15. $\int \frac{x^4 \, d x}{(1 + x)^2}$.

Ans. $\frac{(x + 1)^3}{3} - 2\,(x + 1)^2 + 6\,(x + 1) - 4 \log\,(x + 1) - \frac{\ }{x}$

16. $\int \frac{x^7 \, d x}{\sqrt{a^{16} - x^{16}}}$. *Ans.* $\frac{1}{8} \sin^{-1} \left(\frac{x}{a}\right)^8$.

17. $\int \frac{d x}{x^2\,(1 + x)^2}$. Assume $v = \frac{1}{x} + 1$.

Ans. $2 \log \frac{1 + x}{x} - \frac{1 + 2 x}{x\,(1 + x)}$.

18. $\int \frac{d x}{x \sqrt{2 x - 1}}$. *Ans.* $2 \tan^{-1} \sqrt{(2 x - 1)}$.

19. $\int \frac{d x}{x^2 - 16}$. *Ans.* $\frac{1}{8} \log \left(\frac{x - 4}{x + 4}\right)$.

20. $\int \frac{d x}{x^2 + 11 x + 30}$. *Ans.* $\text{Log} \left(\frac{x + 5}{x + 6}\right)$.

21. $$\int \frac{dx}{x^2 + 2x + 5}. \qquad \textit{Ans.} \quad \frac{1}{2} \tan^{-1}\left(\frac{x+1}{2}\right).$$

22. $$\int \frac{dx}{\sqrt{x^2 + 2x + 5}}.$$

$$\textit{Ans.} \quad \text{Log}\,\{x + 1 + \sqrt{(x^2 + 2x + 5)}\}.$$

23. $$\int \frac{dx}{\sqrt{3 - 2x - x^2}}. \qquad \textit{Ans.} \quad \text{Sin}^{-1}\left(\frac{x+1}{2}\right).$$

24. $$\int \frac{dx}{x\sqrt{(x^2 - 1)}}. \qquad \textit{Ans.} \quad \text{Cos}^{-1}\left(\frac{1}{x}\right).$$

25. $$\int \frac{dx}{x\sqrt{4 + 4x + 3x^2}}.$$

$$\textit{Ans.} \quad \frac{1}{2} \log\left(\frac{x}{2 + x + \sqrt{4 + 4x + 3x^2}}\right).$$

26. $$\int \frac{dx}{(2 + 3x^2)^{\frac{3}{2}}}. \qquad \textit{Ans.} \quad \frac{x}{2(2 + 3x^2)^{\frac{1}{2}}}.$$

27. $$\int \frac{(3 + 4x)\,dx}{5x^2 + 2x + 2}.$$

$$\textit{Ans.} \quad \frac{2}{5} \log(5x^2 + 2x + 2) + \frac{11}{15} \tan^{-1}\left(\frac{5x+1}{3}\right).$$

28. $$\int \frac{(1 + e^x + \cos x - \sin x)\,dx}{x + e^x + \sin x + \cos x}.$$

$$\textit{Ans.} \quad \text{Log}\,(x + e^x + \sin x + \cos x).$$

29. $$\int \sqrt{\frac{a - x}{a + x}}\,dx. \qquad \textit{Ans.} \quad a \sin^{-1}\left(\frac{x}{a}\right) + \sqrt{(a^2 - x^2)}.$$

30. $$\int \frac{dx}{\sqrt{(x + a)} + \sqrt{(x + b)}}.$$

$$\textit{Ans.} \quad \frac{2}{3(a - b)} \{(x + a)^{\frac{3}{2}} - (x + b)^{\frac{3}{2}}\}.$$

31. $$\int \sin^2(ax)\,dx. \qquad \textit{Ans.} \quad \frac{x}{2} - \frac{\sin 2ax}{4a}.$$

32. $\int \cos^2 (a x)\, d x.$ *Ans.* $\dfrac{x}{2} + \dfrac{\sin 2 a x}{4 a}.$

33. $\int \sin^7 x\, d x.$

Ans. $\dfrac{\cos^7 x}{7} - \dfrac{3 \cos^5 x}{5} + \cos^3 x - \cos x.$

34. $\int \cos^5 (a x + b)\, d x.$

Ans. $\dfrac{1}{a}\left\{\dfrac{\sin^5 (a x + b)}{5} - \dfrac{2}{3} \sin^3 (a x + b) + \sin (a x + b)\right\}.$

CHAPTER XII.

INTEGRATION BY PARTS.

ART. 39.—By Art. 5 we have

$$d(vu) = v\,du + u\,dv,$$

where u and v are functions of x; therefore, on integrating and transposing we get

$$\int v\,du = uv - \int u\,dv. \quad . \quad . \quad . \quad \text{(P)}$$

We shall apply this formula in the working out of a few easy examples.

(138.) **To integrate** $x \sin x\,dx$.

Let $u = -\cos x$, and $v = x$; therefore $du = \sin x\,dx$, and $dv = dx$. Hence

$$\int x \sin x\,dx = -x \cos x + \int \cos x\,dx;$$

that is,

$$\int x \sin x\,dx = -x \cos x + \sin x.$$

(139.) **To integrate** $e^{nx} \sin kx\,dx$.

Let $\sin kx = v$ and $\frac{1}{n} e^{nx} = u$; therefore $du = e^{nx}\,dx$, and $dv = k \cos kx\,dx$; therefore

$$\int e^{nx} \sin kx\,dx = \frac{1}{n} e^{nx} \sin kx - \frac{k}{n} \int e^{nx} \cos kx\,dx$$

$$= \frac{1}{n} e^{nx} \sin kx - \frac{k}{n^2} e^{nx} \cos kx - \frac{k^2}{n^2} \int e^{nx} \sin kx\,dx.$$

Therefore, by transposing we have

$$\left(\frac{n^2+k^2}{n^2}\right)\int e^{nx}\sin kx\,dx = \frac{1}{n^2}e^{nx}(n\sin kx - k\cos kx).$$

$$\therefore\quad \int e^{nx}\sin kx\,dx = \frac{e^{nx}(n\sin kx - k\cos kx)}{n^2+k^2}$$

$$= \frac{e^{nx}}{\sqrt{n^2+k^2}}\sin\left\{kx - \tan^{-1}\left(\frac{k}{n}\right)\right\}.$$

In like manner we have

$$\int e^{nx}\cos kx\,dx = \frac{e^{nx}}{\sqrt{n^2+k^2}}\sin\left\{kx + \tan^{-1}\left(\frac{n}{k}\right)\right\}.$$

(140.) **To integrate** $\sqrt{(r^2-x^2)}\,dx$.

Here

$$\int\sqrt{r^2-x^2}\,dx = \int\frac{(r^2-x^2)\,dx}{\sqrt{r^2-x^2}}$$

$$= r^2\int\frac{dx}{\sqrt{r^2-x^2}} - \int\frac{x^2\,dx}{\sqrt{r^2-x^2}}.$$

Therefore

$$\int\sqrt{r^2-x^2}\,dx = r^2\sin^{-1}\left(\frac{x}{r}\right) - \int\frac{x^2\,dx}{\sqrt{r^2-x^2}}. \qquad (a)$$

Again, on integrating by parts we get

$$\int\sqrt{(r^2-x^2)}\,dx = x\sqrt{r^2-x^2} + \int\frac{x^2\,dx}{\sqrt{r^2-x^2}}. \qquad (b)$$

On adding (*a*) and (*b*) and dividing by 2, we get

$$\int\sqrt{(r^2-x^2)}\,dx = \frac{r^2}{2}\sin^{-1}\left(\frac{x}{r}\right) + \frac{x\sqrt{r^2-x^2}}{2}.$$

Similarly, we may show that

$$\int\sqrt{(x^2-r^2)}\,dx = -\frac{r^2}{2}\log\left\{x+\sqrt{(x^2-r^2)}\right\} + \frac{x}{2}\sqrt{(x^2-r^2)}.$$

These two integrals may be used for finding the areas a circle and hyperbola whose equations are $x^2 + y^2 = r^2$ and $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ respectively.

(141.) **To integrate** $x^2 e^x \, d x$.

Let $x^2 = v$ and $e^x = u$; therefore $e^x \, d x = d u$, and on ubstituting in the formula for integrating by parts, we ave

$$\int x^2 e^x \, d x = x^2 e^x - 2 \int x e^x \, d x.$$

Again,

$$\int x e^x \, d x = x e^x - \int e^x \, d x = x e^x - e^x,$$

herefore

$$\int x^2 e^x \, d x = x^2 e^x - 2 x e^x + 2 e^x = e^x (x^2 - 2 x + 2).$$

(142.) **To integrate** $x \tan^{-1} x \, d x$.

Let $\frac{x^2}{2} = u$ and $\tan^{-1} x = v$; therefore $d v = \frac{d x}{1 + x^2}$; and substituting in the formula we have

$$\int x \tan^{-1} x \, d x = \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \frac{x^2 \, d x}{1 + x^2}.$$

Again,

$$\int \frac{x^2 \, d x}{1 + x^2} = \int \left(1 - \frac{1}{1 + x^2}\right) d x = x - \tan^{-1} x;$$

herefore

$$\int x \tan^{-1} x \, d x = \frac{x^2}{2} \tan^{-1} x + \frac{\tan^{-1} x}{2} - \frac{x}{2}.$$

EXAMPLES.

1. $\int \log a x \, d x$. *Ans.* $x \log a x - x$.

2. $\int x^m \log x \, d x$. *Ans.* $\frac{x^{m+1}}{m+1} \left\{ \log x - \frac{1}{m+1} \right\}$.

3. $\int \theta \tan^2 \theta \, d\,\theta.$ *Ans.* $\theta \tan \theta + \log(\cos \theta) - \dfrac{\theta^2}{2}.$

4. $\int \sqrt{\left(\dfrac{a+x}{x}\right)}\, d\,x.$

Ans. $\sqrt{a\,x + x^2} + a \log\{\sqrt{x} + \sqrt{(a+x)}\}.$

5. $\int (1 + \cos x)^2 \, d\,x.$ *Ans.* $\dfrac{3\,x}{2} + 2 \sin x + \dfrac{\sin 2\,x}{4}$

6. $\int e^x \sin p\,x \cos q\,x\, d\,x.$

Ans. $$\frac{e^x}{2}\left\{\frac{\sin(p+q)\,x - (p+q)\cos(p+q)\,x}{1+(p+q)^2}\right.$$
$$\left. + \frac{\sin(p-q)\,x - (p-q)\cos(p-q)\,x}{1+(p-q)^2}\right\}.$$

7. $\int \dfrac{\theta + \sin\theta}{1+\cos\theta}\, d\,\theta.$ *Ans.* $\theta \tan \dfrac{\theta}{2}.$

8. $\int \dfrac{\log(\log\theta)\, d\,\theta}{\theta}.$ *Ans.* $\log\theta \times \log(\log\theta) - \mathrm{lo}$ $\theta.$

9. $\int \sqrt{2\,x - x^2}\, d\,x.$

Ans. $\dfrac{x-1}{2}\sqrt{(2\,x - x^2)} + \dfrac{1}{2}\sin^{-1}(x-1).$

10. $\int \dfrac{d\,x}{1+e^x}.$ *Ans.* $\mathrm{Log}\,\dfrac{e^x}{e^x+1}.$

11. $\int x^n (\log x)^2 \, d\,x.$

Ans. $\dfrac{x^{n+1}}{n+1}\left\{\dfrac{2}{(1+n)^2} - \dfrac{2}{1+n}\log x + (\log x)^2\right\}.$

12. $\int \dfrac{\log x\, d\,x}{(1-x)^2}.$ *Ans.* $\dfrac{x \log x}{1-x} + \log(1-x).$

CHAPTER XIII.

ART. 40.—Integration of Rational Fractions.

Suppose we require the integral of such an expression as

$$\frac{a + b\,x + c\,x^2 \;\; . \;\; . \;\; . \;\; p\,x^{n-1} + x^n}{a' + b'\,x + c'\,x^2. \;\; . \;\; . \;\; p'\,x^{m-1} + x^m}\,d\,x,$$

where $a, b, c \;.\;.\;.\; a', b', c' \;.\;.\;.$ are constants, m and n being positive integers.

If n be greater than m; by division we have a quotient, consisting of a rational function of x, which is integrable by A, Art. (37), and a remainder whose numerator is of a lower degree than the denominator. In order to integrate this remainder we must decompose it into a number of more simple fractions called *partial fractions.*

Denoting the remainder by $\frac{f\,(x)}{\phi\,(x)}$, $\phi\,(x)$ being

$$\begin{aligned} & a' + b'\,x + c'\,x^2 \;\; . \;\; . \;\; . \;\; p'\,x^{m-1} + x^m \\ & = (x - \alpha)\,(x - \beta)\,(x - \gamma) \;\; . \;\; . \;\; . \;\; . \end{aligned}$$

where $\alpha, \beta, \gamma \;.\;.\;.\;.$ are the roots of $\phi\,(x) = 0$ supposed real and unequal; then

$$\begin{aligned} \frac{f\,(x)}{\phi\,(x)} &= \frac{f\,(x)}{(x - \alpha)\,(x - \beta)\,(x - \gamma) \;\; . \;\; . \;\; .} \\ &= \frac{\mathrm{A}}{x - \alpha} + \frac{\mathrm{B}}{x - \beta} + \frac{\mathrm{C}}{x - \gamma} \;\; . \;\; . \;\; . \;\; . \end{aligned}$$

where A, B, C . . . are constants, and therefore independent of x.

Clearing of fractions, we have

$$f(x) = A(x-\beta)(x-\gamma) \;.\;.\;.\;.\; + B(x-\alpha)(x-\gamma) \;.\;.\;.\; + \&c. \quad .\;.\;.\;.\;.\quad (1)$$

This expression holds for all values of x; therefore it holds when $x = \alpha$, $x = \beta$, &c. Therefore

$$f(\alpha) = A(\alpha-\beta)(\alpha-\gamma) \;.\;.\;.\;.$$

Similarly,

$$f(\beta) = B(\beta-\alpha)(\beta-\gamma) \;.\;.\;.\;.$$

$$\&c. \qquad\qquad \&c.$$

It will be noticed that the right-hand member of (1) is $\phi'(x)$.

Therefore

$$f(\alpha) = A\,\phi'(\alpha).$$

Therefore

$$A = \frac{f(\alpha)}{\phi'(\alpha)}.$$

Similarly,

$$B = \frac{f(\beta)}{\phi'(\beta)}.$$

$$C = \frac{f(\gamma)}{\phi'(\gamma)}.$$

$$\&c. \qquad \&c.$$

Therefore the integral of $\frac{f(x)}{\phi(x)}\,dx$ is

$$\frac{f(\alpha)}{\phi'(\alpha)}\int\frac{dx}{x-\alpha} + \frac{f(\beta)}{\phi'(\beta)}\int\frac{dx}{x-\beta} + \&c.$$

$$= \frac{f(\alpha)}{\phi'(\alpha)}\log(x-\alpha) + \frac{f(\beta)}{\phi'(\beta)}\log(x-\beta) + \&c.$$

(143.) **Integrate** $\frac{x^5\,dx}{x^3 - 7x - 6}$.

Here

$$\frac{x^5}{x^3 - 7x - 6} = x^2 + 7 + \frac{6x^2 + 49x + 42}{(x+1)(x+2)(x-3)},$$

and

$$\frac{f(x)}{\phi(x)} = \frac{6x^2 + 49x + 42}{(x+1)(x+2)(x-3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x-3};$$

lso $\phi'(x) = 3x^2 - 7$. Therefore

$$A = \frac{f(-1)}{\phi'(-1)} = \frac{1}{4},$$

$$B = \frac{f(-2)}{\phi'(-2)} = -\frac{32}{5},$$

and

$$C = \frac{f(3)}{\phi'(3)} = \frac{243}{20};$$

therefore

$$\frac{x^5\,dx}{x^3 - 7x - 6} = \int x^2\,dx + \int 7\,dx + \frac{1}{4}\int\frac{dx}{x+1}$$

$$-\frac{32}{5}\int\frac{dx}{x+2} - \frac{243}{20}\int\frac{dx}{x-3} = \frac{x^3}{3} + 7x + \frac{1}{4}\log(x+1)$$

$$-\frac{32}{5}\log(x+2) + \frac{243}{20}\log(x-3).$$

ART. 41.—Suppose we require the integral of the rational fraction $\frac{f(x)}{\phi(x)}$ where $\phi(x)$ has n equal roots; then

$$\frac{f(x)}{\phi(x)} = \frac{f(x)}{(x-a)^n\,F(x)}$$

where a is one of the n equal roots.

Assume

$$\frac{f(x)}{(x-a)^n\,F(x)} = \frac{A}{(x-a)^n} + \frac{B}{(x-a)^{n-1}} \cdots \frac{P}{x-a} + \frac{Q}{F(x)},$$

and, clear of fractions, we have

$$f(x) = A\,F(x) + B\,(x - a)\,F(x) + \ldots\ Q\,(x - a)^n \quad . \ . \ (1)$$

and since $x = a$, we have

$$f(a) = A\,F(a),$$

since the remaining terms on the right-hand side vanish; therefore

$$A = \frac{f(a)}{F(a)}.$$

To get B, differentiate (1); therefore

$$f'(x) = A\,F'(x) + B\,(x - a)\,F'(x) + B\,F(x) \\ + C\,(x - a)^2\,F'(x) + 2\,C\,(x - a)\,F(x) + \&c.$$

Substituting a for x, we get

$$f'(a) = A\,F'(a) + B\,F(a).$$

The remaining terms vanish when $x = a$; therefore

$$B = \frac{f'(a) - A\,F'(a)}{F(a)}.$$

By differentiating (1) a second time, and substituting a for x, we can find C, and by repeating this process we can find all the constants A, B, C . . . P.

(144.) **Integrate** $\dfrac{x^2\,dx}{(x-1)^2\,(x+2)\,(x-3)}$.

Assume

$$\frac{x^2}{(x-1)^2\,(x+2)\,(x-3)} = \frac{A}{(x-1)^2} + \frac{B}{(x-1)} \\ + \frac{C}{x+2} + \frac{D}{x-3},$$

and by the method shown above it will be found that

$$A = -\frac{1}{6}, \quad B = -\frac{13}{36}, \quad C = -\frac{4}{45}, \quad \text{and} \quad D = \frac{9}{20};$$

therefore

$$\int \frac{x^2\,dx}{(x-1)^2(x+2)(x-3)} = -\frac{1}{6}\int\frac{dx}{(x-1)^2} - \frac{13}{16}\int\frac{dx}{x-1}$$

$$-\frac{4}{45}\int\frac{dx}{x+2} + \frac{9}{20}\int\frac{dx}{x-3}$$

$$= \frac{1}{6(x-1)} - \frac{13}{16}\log(x-1) - \frac{4}{45}\log(x+2) + \frac{9}{20}\log(x-3).$$

ART. 42.—Suppose we require the integral of $\frac{f(x)}{\phi(x)}\,dx$ where $\phi(x)$ has one pair of imaginary roots.

The quadratic factor corresponding to the pair of imaginary roots is of the form $x^2 + ax + b$.

Assume

$$\frac{f(x)}{\phi(x)} = \frac{Ax + B}{x^2 + ax + b} + \frac{C}{x - \alpha} + \frac{D}{x - \beta} + \&c.$$

Multiplying by $\phi(x)$ we have

$$f(x) = (Ax + B)(x - \alpha)(x - \beta) \;.\;.\;.\;.$$
$$+ C(x^2 + ax + b)(x - \beta) \;.\;.\;.\;. + \&c.$$

The values of A, B, C, D, &c., may be found from this identity by equating the coefficients of like powers of x.

(145.) **Integrate** $\frac{x^2\,dx}{(x^2+2)(x-1)^2}$.

Assume

$$\frac{x^2}{(x^2+2)(x-1)^2} = \frac{Ax + B}{x^2 + 2} + \frac{C}{(x-1)^2} + \frac{D}{x-1}.$$

Clearing of fractions, and equating coefficients of like powers of x, we get

$$A = -\frac{4}{9}, \quad B = \frac{2}{9}, \quad C = \frac{1}{3}, \quad \text{and} \quad D = \frac{4}{9};$$

therefore

$$\int\frac{x^2\,dx}{(x^2+2)(x-1)^2} = -\frac{2}{9}\int\frac{(2x-1)\,dx}{x^2+2} + \frac{1}{3}\int\frac{dx}{(x-1)^2}$$

$$+\frac{4}{9}\int\frac{d\,x}{x-1}=-\frac{2}{9}\log\left(x^2+2\right)+\frac{\sqrt{2}}{9}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)$$

$$-\frac{1}{3\,(x-1)}+\frac{4}{9}\log\,(x-1).$$

(146.) **Integrate** $\dfrac{d\,x}{(x^2+x+1)\,(x^2+1)}$.

Assume

$$\int\frac{1}{(x^2+x+1)\,(x^2+1)}=\frac{A\,x+B}{x^2+x+1}+\frac{C\,x+D}{x^2+1}.$$

Clearing of fractions, and equating the coefficients of like powers of x, we get

$$A=1,\quad B=1,\quad C=-1,\quad \text{and}\quad D=0;$$

therefore

$$\int\frac{d\,x}{(x^2+x+1)\,(x^2+1)}=\int\frac{(x+1)\,d\,x}{x^2+x+1}-\int\frac{x\,d\,x}{x^2+1}$$

$$=\frac{1}{2}\log\left(x^2+x+1\right)+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\,x+1}{\sqrt{3}}\right)-\frac{1}{2}\log\left(x^2+1\right)$$

$$=\frac{1}{2}\log\left(\frac{x^2+x+1}{x^2+1}\right)+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\,x+1}{\sqrt{3}}\right).$$

(147.) **Integrate** $\dfrac{x\,d\,x}{x^4+x^2-2}$.

Assume

$$\frac{x}{x^4+x^2-2}=\frac{A\,x+B}{x^2+2}+\frac{C}{x+1}+\frac{D}{x-1}.$$

Clearing of fractions, and equating coefficients of like powers of x, we get

$$A=-\frac{1}{3},\quad B=0,\quad C=\frac{1}{6},\quad \text{and}\quad D=\frac{1}{6};$$

therefore

$$\int \frac{x\,dx}{x^4 + x^2 - 2} = -\frac{1}{3}\int \frac{x\,dx}{x^2+2} + \frac{1}{6}\int \frac{dx}{x+1} + \frac{1}{6}\int \frac{dx}{x-1}$$

$$= -\frac{1}{6}\log(x^2+2) + \frac{1}{6}\log(x+1) + \frac{1}{6}\log(x-1)$$

$$= \frac{1}{6}\log\left(\frac{x^2-1}{x^2+2}\right).$$

This example admits of an easier solution by assuming $x^2 = z$.

EXAMPLES.

1. $\displaystyle\int \frac{dx}{1-x^3}$.

Ans. $\displaystyle\frac{1}{6}\log(1+x+x^2) - \frac{1}{3}\log(1-x)$

$$+ \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right).$$

2. $\displaystyle\int \frac{(x-1)\,dx}{x^2-4}$. *Ans.* $\log(x+2)^{\frac{3}{4}}(x-2)^{\frac{1}{4}}$.

3. $\displaystyle\int \frac{x^3\,dx}{x^2+6x+8}$.

Ans. $\displaystyle\frac{x^2}{2} - 6x + 32\log(x+4) - 4\log(x+2)$.

4. $\displaystyle\int \frac{(x^2-2x+3)\,dx}{x^2-3x+2}$.

Ans. $x + 3\log(x-2) - 2\log(x-1)$.

5. $\displaystyle\int \frac{x\,dx}{x^3+x^2+x+1}$.

Ans. $\displaystyle\frac{1}{2}\tan^{-1}x + \frac{1}{4}\log(1+x^2) - \frac{1}{2}\log(1+x)$.

6. $\int \frac{(x^2+1)\,d\,x}{x^4+x^2+1}$.

Ans. $\frac{1}{\sqrt{3}}\tan^{-1}\frac{2\,x+1}{\sqrt{3}}+\frac{1}{\sqrt{3}}\tan^{-1}\frac{2\,x-1}{\sqrt{3}}$.

7. $\int \frac{d\,x}{(1+x)(1+x^2)(1+2\,x)}$.

Ans. $\frac{4}{5}\log(2\,x+1)-\frac{3}{20}\log(x^2+1)$

$-\frac{1}{10}\tan^{-1}x-\frac{1}{2}\log(x+1)$.

8. $\int \frac{x\,d\,x}{x^2-x-2}$. *Ans.* $\frac{2}{3}\log(x-2)+\frac{1}{3}\log(x+1)$.

9. $\int \frac{x^2\,d\,x}{x^4+3\,x^2+2}$. *Ans.* $\sqrt{2}\tan^{-1}\frac{x}{\sqrt{2}}-\tan^{-1}x$.

10. $\int \frac{(3\,x-4)\,d\,x}{(x+2)(x+1)^2}$.

Ans. $10\log\frac{x+1}{x+2}+\frac{7}{x+1}$.

11. $\int \frac{x^2\,d\,x}{x^4+4\,x^3+5\,x^2+4\,x+4}$.

Ans. $\frac{2}{25}\log(1+x^2)-\frac{4}{25}\log(x+2)$

$-\frac{4}{5(x+2)}-\frac{3}{25}\tan^{-1}x$.

12. $\int \frac{d\,x}{(1-x^3)^{\frac{1}{3}}}$. Assume $1-x^3=y^3x^3$.

Ans. $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2\,y-1}{\sqrt{3}}\right)-\log(1+y)$

where $y^3=\frac{1-x^3}{x^3}$.

13. $$\int \frac{x^3\,d\,x}{x^4+1}.\qquad \textit{Ans.}\quad \frac{1}{4}\log(x^4+1).$$

14. $$\int \frac{(1+x^2)\,d\,x}{x(1+x^2+x^4)}.$$

$$\textit{Ans.}\quad \operatorname{Log} x - \frac{1}{4}\log(x^2-x+1) + \frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{2\,x-1}{\sqrt{3}}\right)$$
$$-\frac{1}{4}\log(x^2+x+1) - \frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{2\,x+1}{\sqrt{3}}\right).$$

15. $$\int \frac{d\,x}{(x^2+p^2)(x+q)}.$$

$$\textit{Ans.}\quad \frac{1}{p^2+q^2}\left\{\log(x+q) - \frac{1}{2}\log(x^2+p^2)\right.$$
$$\left. + \frac{q}{p}\tan^{-1}\left(\frac{x}{p}.\right)\right\}.$$

16. $$\int \frac{\theta\,d\,\theta}{\theta^6+1}.$$

$$\textit{Ans.}\quad \frac{1}{2\sqrt{3}}\{\tan^{-1}(2\theta-\sqrt{3}) - \tan^{-1}(2\,\theta+\sqrt{3})\}$$
$$+\frac{1}{12}\log\frac{1-\theta^2+\theta^4}{(1+\theta^2)^2}.$$

CHAPTER XIV.

ART. 43.—Integration by Successive Reduction.

Suppose we require the integral of $\sin^n x\,d\,x$ where n is positive integer. If n be odd, the required integral is easil obtained; for let $z = \cos x$; therefore $d\,z = -\sin x\,d\,x$, an $\sin^n x\,d\,x = \sin^{n-1} x \sin x\,d\,x = -\sin^{n-1} x\,d\,(\cos x)$, and su stituting in terms of z we get $\sin^n x\,d\,x = -(1 - z^2)^{\frac{n-1}{2}}\,d$ but $\frac{n-1}{2}$ is an even integer if n be greater than unity and after expanding by the Binomial Theorem the integr is easily obtained.

Similarly, if the integral of $\cos^n x\,d\,x$ is required where is an odd positive integer, and greater than unity, let $z = \sin x$ $\therefore\ d\,z = \cos x\,d\,x$, also $\cos^n x\,d\,x = \cos^{n-1} x \cos x\,d\,x =$ $\cos^{n-1} x\,d\,(\sin x)$, therefore by substitution

$$\cos^n x\,d\,x = (1 - z^2)^{\frac{n-1}{2}}\,d\,z$$

which is easily integrable since $\frac{n-1}{2}$ is an even positive integer.

(148.) **Integrate** $\sin^7 x\,d\,x$.

Here, $\sin^7 x\,d\,x = \sin^6 x \sin x\,d\,x = -\sin^6 x\,d\,(\cos x$ $= -(1 - z^2)^3\,d\,z$, where $z = \cos x$, therefore

$$\int \sin^7 x\,d\,x = -\int (1 - z^2)^3\,d\,z = -\int (1 - 3\,z^2 + 3\,z^4 - z^6)\,d\,z$$

$$= -z + z^3 - \frac{3}{5} z^5 + \frac{z^7}{7}$$

$$= -\cos x + \cos^3 x - \frac{3}{5} \cos^5 x + \frac{1}{7} \cos^7 x.$$

(149.) **Integrate** $\cos^5 x\,d\,x$.

Here, $\cos^5 x\,d\,x = \cos^4 x \cos x\,d\,x = (1 - \sin^2 x)^2\,d\,(\sin x)$ $= (1 - z^2)\,d\,z$ where $z = \sin x$, therefore

$$\int \cos^5 x\,d\,x = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x.$$

ART. 44.—Suppose we require the integral of $\sin^n x\,d\,x$ where n is an even positive integer.

We have

$$\int \sin^n x\,d\,x = \int \sin^{n-1} x \sin x\,d\,x = -\int \sin^{n-1} x\,d\,(\cos x).$$

Integrating by parts we have

$$\int \sin^n x\,d\,x = -\cos x \sin^{n-1} x + (n-1)\int \cos^2 x \sin^{n-2} x\,d\,x$$
$$= -\cos x \sin^{n}{}_{-1} x + (n-1)\int (1 - \sin^2 x)\sin^{n-2} x\,d\,x,$$

and by transposing and dividing by n we have

$$\int \sin^n x\,d\,x = -\frac{\cos x \sin^{n-1} x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,d\,x.$$

By successive application of this result to the integral on the right-hand side, we get

$$\int \sin^n x\,d\,x = -\frac{\cos x}{n}\left\{\sin^{n-1} x + \frac{n-1}{n-2}\sin^{n-3} x + \frac{(n-1)(n-3)}{(n-2)(n-4)}\sin^{n-5} x \ \ldots\right\}$$
$$+ \frac{(n-1)(n-3)\ \ldots\ 1}{(n-2)(n-4)\ \ldots\ 2}\,x. \quad \ldots \quad \text{A.}$$

Similarly, it can be shown that

$$\int \cos^n x\,d\,x = \frac{\sin x}{n}\left\{\cos^{n-1} x + \frac{n-1}{n-2}\cos^{n-3} x + \frac{(n-1)(n-3)}{(n-2)(n-4)}\cos^{n-5} x \ \ldots\right\}$$
$$+ \frac{(n-1)(n-3)(n-5)\ \ldots\ 1}{n(n-2)(n-4)\ \ldots\ldots\ 2}\,x \quad \ldots \quad \text{B.}$$

(150.) **Integrate** $\sin^2 x\,d\,x$.

$$\int \sin^2 x\,d\,x = \int \frac{(1 - \cos 2\,x)\,d\,x}{2} = \frac{1}{2}\int d\,x - \frac{1}{2}\int \cos 2\,x\,d\,x;$$

therefore

$$\int \sin^2 x\,d\,x = \frac{x}{2} - \frac{\sin 2\,x}{4}.$$

(151.) **Integrate** $\cos^2 x\,d\,x$.

$$\int \cos^2 x\,d\,x = \int \frac{(1 + \cos 2\,x)\,d\,x}{2} = \frac{1}{2}\int d\,x + \frac{1}{2}\int \cos 2\,x\,d\,x;$$

therefore

$$\int \cos^2 x\,d\,x = \frac{x}{2} + \frac{\sin 2\,x}{4}.$$

(152.) **Integrate** $\sin^4 x\,d\,x$.

By A, we have

$$\int \sin^4 x\,d\,x = -\frac{\cos x \sin^3 x}{4} + \frac{3}{4}\int \sin^2 x\,d\,x.$$

$$= -\frac{\cos x \sin^3 x}{4} + \frac{3\,x}{8} - \frac{3}{16}\sin 2\,x, \text{ by 150.}$$

(153.) **Integrate** $\cos^4 x\,d\,x$.

By B, we have

$$\int \cos^4 x\,d\,x = \frac{\sin x \cos^3 x}{4} + \frac{3}{4}\int \cos^2 x\,d\,x$$

$$= \frac{\sin x \cos^3 x}{4} + \frac{3\,x}{8} + \frac{3}{16}\sin 2\,x, \text{ by 151.}$$

Art. 45.—Suppose we require the integral of

$$\sin^n x \cos^m x\,d\,x$$

where n and m are both positive integers.

First let m be odd. The expression may be written thus:

$$\int \sin^n x \cos^{m-1} x \cos x\,d\,x = \int \sin^n x \cos^{m-1} x\,d\,(\sin x).$$

Substitute z for $\sin x$ and we have

$$\int \sin^n x \cos^m x \, d\, x = \int z^n (1 - z^2)^{\frac{m-1}{2}} d\, z,$$

and since $\frac{m-1}{2}$ is an even integer, the expression is easily integrated after expanding by the binomial theorem.

Again, suppose n and m both positive and even.

Let $v = \frac{\sin^{n+1} x}{n+1}$ and $u = \cos^{m-1} x$;

therefore $d\, v = \sin^n x \cos x \, d\, x$ and integrating by parts we have

$$\int \sin^n x \cos^m x \, d\, x = \frac{\sin^{n+1} x \cos^{m-1} x}{n+1}$$
$$+ \frac{m-1}{n+1} \int \sin^{n+2} x \cos^{m-2} x \, d\, x.$$

By repeated application of this formula and by aid of A, Art. 44, the required integral can be obtained.

(154.) **To integrate** $e^{a x} x^n \, d\, x$.

Integrating by parts we have

$$\int e^{a x} x^n \, d\, x = \frac{1}{a} e^{a x} x^n - \frac{n}{a} \int e^{a x} x^{n-1} \, d\, x$$
$$= \frac{1}{a} e^{a x} x^n - \frac{n}{a^2} e^{a x} x^{n-1} + \frac{n(n-1)}{a^2} \int e^{a x} x^{n-2} \, d\, x, \text{ \&c.}$$

(155.) **To integrate** $e^{a x} \sin^n x \, d\, x$.

Integrating by parts we have

$$\int e^{a x} \sin^n x \, d\, x = \frac{1}{a} e^{a x} \sin^n x - \frac{n}{a} \int e^{a x} \sin^{n-1} x \cos x \, d\, x$$
$$= \frac{1}{a} e^{a x} \sin^n x - \frac{n}{a^2} e^{a x} \sin^{n-1} x \cos x$$
$$+ \frac{n}{a^2} \int e^{a x} \{(n-1) \sin^{n-2} x (1 - \sin^2 x) - \sin^n x\} \, d\, x.$$

Transposing and dividing by $a^2 + n^2$ we have

$$\int e^{ax} \sin^n x\, dx = \frac{e^{ax}}{a^2 + n^2} \sin^{n-1} x\,(a \sin x - n \cos x)$$
$$+ \frac{n(n-1)}{a^2 + n^2} \int e^{ax} \sin^{n-2} x\, dx.$$

Repeating this process on the integral on the right-hand side we, finally, obtain the required integral.

(156.) **To integrate** $\frac{dx}{\sin^n x}$ **and** $\frac{dx}{\cos^n x}$.

$$\int \frac{dx}{\sin^n x} = -\frac{\cos x}{(n-1)\sin^{n-1} x} + \frac{n-2}{n-1} \int \frac{dx}{\sin^{n-2} x}$$
$$= -\frac{\cos x}{(n-1)\sin^{n-1} x} - \frac{(n-2)}{(n-1)} \frac{\cos x}{(n-3)\sin^{n-3} x}$$
$$+ \frac{(n-2(n-4)}{(n-1(n-3)} \int \frac{dx}{\sin^{n-4} x}, \text{ \&c.}$$

Similarly,

$$\int \frac{dx}{\cos^n x} = \frac{\sin x}{(n-1)\cos^{n-1} x} + \frac{(n-2)(n-4)\sin x}{(n-1)(n-3)\cos^{n-3} x}$$
$$+ \frac{(n-2)(n-4)}{(n-1)(n-3)} \int \frac{dx}{\cos^{n-4} x}, \text{ \&c.}$$

EXAMPLES.

1. $\int \sin^3 x\, dx$. *Ans.* $\frac{\cos^3 x}{3} - \cos x$.

2. $\int \sin^5 x\, dx$. *Ans.* $-\frac{\cos^5 x}{5} + \frac{2}{3}\cos^3 x - \cos x$.

3. $\int \cos^3 x\, dx$. *Ans.* $-\frac{\sin^3 x}{3} + \sin x$.

4. $\int \cos^5 x\, dx$. *Ans.* $\frac{\sin^5 x}{5} - \frac{2}{3}\sin^3 x + \sin x$.

5. $\int \sin^6 x\, d\, x.$

Ans. $$-\frac{\cos x \sin^5 x}{6} - \frac{5}{24}\cos x \sin^3 x - \frac{5}{16}\cos x \sin x + \frac{5\, x}{16}.$$

6. $\int \cos^6 x\, d\, x.$

Ans. $$\frac{\text{Sin}\, x \cos^5 x}{6} - \frac{5 \sin x \cos^3 x}{24} + \frac{5 \sin x \cos x}{16} + \frac{5\, x}{16}.$$

7. $\int \sin x \cos^3 x\, d\, x.$ *Ans.* $\frac{\text{Sin}^2 x}{2} - \frac{\sin^4 x}{4}.$

8. $\int \sin^2 x \cos^5 x\, d\, x.$ *Ans* $\frac{\text{Sin}^3 x}{3} - \frac{2}{5}\sin^5 x + \frac{\sin^7 x}{7}.$

9. $\int \sin^4 x \cos^7 x\, d\, x.$

Ans. $$\frac{\text{Sin}^5 x}{5} - \frac{3}{7}\sin^7 x + \frac{1}{3}\sin^9 x - \frac{1}{11}\sin^{11} x.$$

10. $\int \sin x \cos^2 x\, d\, x.$ *Ans.* $-\frac{\text{Cos}^3 x}{3}.$

11. $\int \sin^3 x \cos^4 x\, d\, x.$ *Ans.* $\frac{\text{Cos}^7 x}{7} - \frac{\cos^5 x}{5}.$

12. $\int \sin^5 x \cos^6 x\, d\, x.$

Ans. $$-\frac{\cos^7 x}{7} + \frac{2}{9}\cos^9 x - \frac{1}{11}\cos^{11} x.$$

13. $\int \sin^2 x \cos^2 x\, d\, x.$ *Ans.* $\frac{x}{8} - \frac{\sin 4\, x}{32}.$

14. $\int \sin^4 x \cos^2 x\, d\, x.$

Ans. $$\frac{\text{Sin}^5 x \cos x}{6} - \frac{\sin^3 x \cos x}{24} - \frac{\sin 2\, x}{32} + \frac{x}{16}.$$

15. $\int \sin^2 x \cos^4 x\, d\, x.$

Ans. $$-\frac{\sin x \cos^5 x}{6} + \frac{1}{24}\sin x \cos^3 x + \frac{1}{16}\sin x \cos x + \frac{x}{16}.$$

16. $\int \sin^4 x \cos^4 x \, d\, x.$

$$Ans. \left\{ \begin{array}{l} -\dfrac{\sin^3 x \cos^5 x}{8} - \dfrac{1}{16} \sin x \cos^5 x + \dfrac{\sin x \cos^3 x}{64} \\ \qquad\qquad + \dfrac{3}{128} \sin x \cos x + \dfrac{3\, x}{128}. \end{array} \right\}$$

17. $\int \tan^6 x \, d\, x.$ *Ans.* $\dfrac{\tan^5 x}{5} - \dfrac{\tan^3 x}{3} + \tan x - x.$

Here,

$$\begin{aligned} \int \tan^6 x \, d\, x &= \int \tan^4 x \, (\sec^2 x - 1) \, d\, x \\ &= \int \tan^4 x \, d\, (\tan x) - \int \tan^4 x \, d\, x, \text{ \&c.} \end{aligned}$$

18. $\int e^{a\, x} x^4 \, d\, x.$

Ans. $\dfrac{e^{a\, x}}{a} \left(x^4 - \dfrac{4\, x^3}{a} + \dfrac{4 \times 3\, x^2}{a^2} - \dfrac{4 \times 3 \times 2\, x}{a^3} + \dfrac{\underline{|4}}{a^4} \right)$

19. $\int e^{3\, x} \sin^3 x \, d\, x.$

Ans. $\dfrac{e^{3\, x}}{6} \left\{ \sin^2 x \, (\sin x - \cos x) + \dfrac{1}{5} (3 \sin x - \cos x) \right\}.$

20. $\displaystyle\int \frac{d\, x}{\sin^5 x}.$

Ans. $-\dfrac{\cos x}{4 \sin^4 x} - \dfrac{3}{8} \dfrac{\cos x}{\sin^2 x} + \dfrac{3}{8} \log \tan \dfrac{x}{2}.$

21. $\displaystyle\int \frac{d\, x}{\cos^3 x}.$ *Ans.* $\dfrac{\sin x}{2 \cos^2 x} + \dfrac{1}{4} \log \left(\dfrac{1 + \sin x}{1 - \sin x} \right).$

CHAPTER XV.

DEFINITE INTEGRALS.

Art. 46.—Hitherto we have confined our attention to indefinite integrals in which no limits were assigned to the variable. We shall now endeavour to give an elementary ea of a definite integral, rerding the process of integraon as a summation.

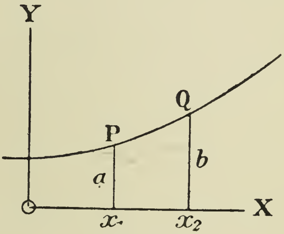


Fig. 16.

Suppose y, the ordinate of the curve P Q, Fig. 16, to vary continuously by successive increments, being a at the point P, and b at the point Q. It is evident that the total increment of y is $b - a$, and considering this total increment as the sum of its partial increments, we have

$$\int_a^b dy = b - a,$$

b being called the superior limit and a the inferior limit of y.

Again, suppose y a function of x, the relation being

$$y = \phi(x),$$

and let $x = x_1$ where $y = a$, and $x = x_2$ where $y = b$; therefore $a = \phi(x_1)$ and $b = \phi(x_2)$.

Also

$$dy = \phi'(x)\, dx;$$

therefore

$$\int_{x_1}^{x_2} \phi'(x)\, dx = \phi(x_2) - \phi(x_1).$$

x_2 is the superior limit and x_1 the inferior limit of x.

We see that, in order to integrate a continuous function of x between certain limits we first find the indefinite integral, and then substitute for x the superior limit and also the inferior limit in the indefinite integral, and subtract the latter from the former.

(157.) **Find the value of** $\int x^3\,dx$ between the limits $x = 2$ and $x = 4$.

$$\int_2^4 x^3\,dx = \left[\frac{x^4}{4}\right]_2^4 = \frac{4^4}{4} - \frac{2^4}{4} = 64 - 4;$$

therefore

$$\int_2^4 x^3\,dx = 60.$$

(158.) **Find the value of the definite integral** $\int_0^4 \frac{x\,dx}{1+x^2}$.

The integral of $\frac{x\,dx}{1+x^2}$ is $\frac{1}{2}\log(1+x^2)$.

Substitute 4 for x, and also 0 for x, and subtract the latter from the former; therefore

$$\int_0^4 \frac{x\,dx}{1+x^2} = \frac{1}{2}\left\{\log 17 - \log 1\right\} = \frac{1}{2}\log 17.$$

The complete discussion of definite integrals is beyond the scope of this elementary treatise.

EXAMPLES.

1. $\int_0^2 x^n\,dx$. *Ans.* $\frac{2^{n+1}}{n+1}$.

2. $\int_1^2 \log x\,dx$. *Ans.* $2\log_e 2 - 1$.

3. $\int_0^\pi \sin x\,dx$. *Ans.* 2.

4. $\int_0^{\frac{\pi}{2}} \cos x\,dx.$ *Ans.* $1.$

5. $\int_0^{\pi} \sin^2 x\,dx.$ *Ans.* $\frac{\pi}{2}.$

6. $\int_0^{\frac{\pi}{2}} \cos^2 x\,dx.$ *Ans.* $\frac{\pi}{4}.$

7. $\int_0^1 \frac{dx}{1+x^2}.$ *Ans.* $\frac{\pi}{4}.$

8. $\int_0^1 (x^3 - 3x^2 + 2x + 1)\,dx.$ *Ans.* $\frac{5}{4}.$

9. $\int_2^3 \frac{x\,dx}{1+x^2}.$ *Ans.* $\text{Log}\,\sqrt{2}.$

10. $\int_0^{\frac{\pi}{2}} \frac{\cos x\,dx}{1+\sin x}.$ *Ans.* $\text{Log}\,2.$

11. $\int_0^a \frac{x^7\,dx}{\sqrt{a^{16}-x^{16}}}.$ *Ans.* $\frac{\pi}{16}.$

12. $\int_{\frac{1}{2}}^1 \frac{dx}{x\sqrt{2x-1}}.$ *Ans.* $\frac{\pi}{2}.$

13. $\int_6^{16} \frac{dx}{x^2-16}.$ *Ans.* $\frac{1}{8}\log 3.$

14. $\int_0^a \sqrt{\frac{a-x}{a+x}}\,dx.$ *Ans.* $\frac{\pi a}{2} - a.$

15. $\int_0^r \sqrt{r^2-x^2}\,dx.$ *Ans.* $\frac{\pi r^2}{4}.$

CHAPTER XVI.

AREAS OF PLANE CURVES.

ART. 47.—Suppose we require the area bounded by tl curve A B, the axis of X, and the ordinates P a, Q b, being rectangular. Let the ordinate at the po and the ordinate a indefinitely near to $y^2 = 4ax$, the abscissa of C, x, $x + dx$.

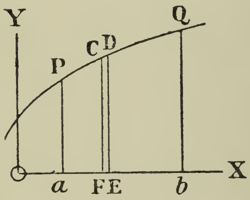


FIG. 17.

Then the area C D $\frac{1}{2}(y + y + dy)\,dx = d\,A$, w is the area required, that is

$$d\,A = y\,dx + \tfrac{1}{2}\,dy\,dx.$$

But when dy and dx are indefinitely small, the term $\frac{1}{2}\,dy\,dx$ is indefinitely small in comparison with $y\,dx$, and may therefore be neglected; therefore

$$d\,A = y\,dx.$$

Integrating we get

$$A = \int y\,dx + C,$$

where C is a constant.

Again, suppose y and x connected by the equation $y = \phi(x)$. Therefore

$$A = \int \phi(x)\,dx + C.$$

But we want the area between the limits $x = a$ and $x = b$; therefore

$$A = \int_a^b \phi(x)\,dx = \Big[f(x)\Big]_a^b,$$

where $f(x)$ is the integral of $\phi(x)\,dx$. Therefore

$$A = f(b) - f(a).$$

The constant C disappears, because when integrating ween limits the constant C is always subtracted from lf.

[illegible] general formula for the area bounded by a con-[illegible]e and the axis of X is

$$A = \int y\,dx + C. \quad . \quad . \quad . \quad . \quad (A)$$

[illegible] formula for the area bounded by any con-[illegible] and the axis of Y is

$$A = \int x\,dy + C. \quad . \quad . \quad . \quad . \quad (B)$$

The axes in both cases being rectangular.

(159.) Find the area bounded by the curve whose equation is $y = x^2$, and the axis of X, between the limits $x = 1$ and $x = 4$.

By formula (A) we have

$$A = \int y\,dx + C = \int_1^4 x^2\,dx + C,$$

that is,

$$A = \left[\frac{x^3}{3} + C\right]_1^4.$$

$$\therefore \quad A = \left(\frac{4^3}{3} + C\right) - \left(\frac{1^3}{3} + C\right) = \frac{64}{3} - \frac{1}{3}.$$

$$\therefore \quad A = 21.$$

(160.) Find the area bounded by the curve mentioned in the last example and the axis of Y between the same limits, viz. $x = 1$ and $x = 4$.

By formula (B) we have

$$A = \int x\,dy + C, \text{ but } dy = 2\,x\,dx.$$

$$\therefore\ A = \int 2\,x^2\,dx + C; \text{ that is,}$$

$$A = \left[\frac{2\,x^3}{3} + C\right]_1^4.$$

$$\therefore\ A = \left(\frac{2 \times 4^3}{3} + C\right) - \left(\frac{2 \times 1^3}{3} + C\right),$$

$$\therefore\ A = 42.$$

(161.) **Find the area of the parabola $y^2 = 4\,a\,x$, between the limits $x = 0$ and $x = a$.**

By (A) we have

$$A = \int y\,dx + C = 2\sqrt{a}\int\sqrt{x}\,dx + C; \text{ that is,}$$

$$A = 2\sqrt{a} \times \frac{2}{3}\,x^{\frac{3}{2}} = \frac{4\sqrt{a\,x} \times x}{3} = \frac{2}{3}\,x\,y.$$

Therefore the required area

$$A = \frac{4}{3}\,a^2.$$

The area of a segment of a parabola bounded by a chord perpendicular to the axis of X is $\frac{2}{3}$ of the area of the circumscribing rectangle.

(162.) **Find the area of a circle whose radius is r.**

If the centre of the circle, Fig. 18, be taken as origin, its equation is $x^2 + y^2 = r^2$; therefore $y = \sqrt{r^2 - x^2}$.

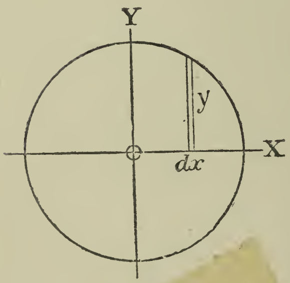


Fig. 18.

By (A), $\frac{1}{4}$ area of the circle,

$$A = \int y\,dx + C = \int_0^r \sqrt{r^2 - x^2}\,dx + C;$$

that is,

$$A = \left[\frac{x\sqrt{r^2 - x^2}}{2} + \frac{r^2}{2}\sin^{-1}\left(\frac{x}{r}\right) + C\right]_0^r$$

by Example (140).

$$\therefore\quad A = \frac{r^2}{2} \times \frac{\pi}{2} = \frac{\pi r^2}{4}.$$

Therefore the whole area of the circle is $4A = \pi r^2$.

(163.) **Find the area of the ellipse whose equation is** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Here

$$y = \frac{b}{a}\sqrt{(a^2 - x^2)},$$

and $\frac{1}{4}$ the required area is

$$A = \frac{b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx;$$

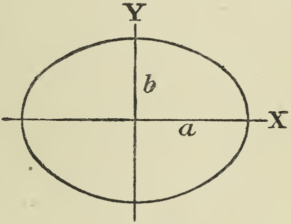


FIG. 19.

that is,

$$A = \frac{b}{a}\left[\frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a;$$

that is,

$$A = \frac{b}{a} \times \frac{\pi a^2}{4} = \frac{\pi a b}{4}.$$

Therefore the whole area of the ellipse is $4A = \pi a b$

(164.) **Find the area of the cycloid**

$$\left.\begin{aligned} x &= a(\theta - \sin\theta) \\ y &= a(1 - \cos\theta) \end{aligned}\right\}\ \text{Art. 34.}$$

The area $A = \int y\,dx + C$; that is,

$$A = a^2 \int_0^{2\pi} (1 - \cos\theta)^2 \,d\theta = a^2 \int (1 - 2\cos\theta + \cos^2\theta)$$

$$= a^2 \left[\theta - 2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4} \right]_0^{2\pi}.$$

$$\therefore \quad A = 3\pi a^2.$$

Therefore the area of a cycloid is three times the area o generating circle.

ART. 48.—The Catenary.

The catenary is the curve assumed by an inelastic s or chain of uniform density suspended by its extremities

We shall investigate the equation of the catenary.

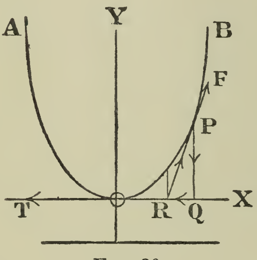


FIG. 20.

Let A O P B (Fig. 20), be catenary.

Taking the vertex as ori O Y and O X as the axis of Y a X respectively.

Consider the equilibrium of t arc O P.

There are three forces acti on it, viz. the horizontal tension T at the vertex, the tangential forc F, at P, and the weight of O which we shall denote by S, assuming unit length to be unit weight. These three forces meet at a point, and a portional to the three sides of the triangle P Q R; that

$$F : T : S :: PR : RQ : QP.$$

Also

$$F = \sqrt{T^2 + S^2}.$$

$$\frac{d\,y}{d\,S} = \frac{S}{F} = \frac{S}{\sqrt{T^2 + S^2}}.$$

$$\therefore \quad y = \int \frac{S\,d\,S}{\sqrt{T^2 + S^2}} + C,$$

ere C is a constant.

$$\therefore \quad y = \sqrt{T^2 + S^2} + C.$$

t $y = 0$ where $S = 0$; therefore $C = -T$. Therefore

$$y + T = \sqrt{T^2 + S^2}. \quad . \quad . \quad . \quad (A)$$

We have also

$$\frac{d\,x}{d\,S} = \frac{T}{\sqrt{T^2 + S^2}}.$$

erefore

$$x = T \int \frac{d\,S}{\sqrt{T^2 + S^2}} + C;$$

at is,

$$x = T \log \{S + \sqrt{T^2 + S^2}\} + C,$$

nd where $x = 0$, $S = 0$, therefore $C = -T \log T$, therefore

$$x = T \log \left\{\frac{S + \sqrt{T^2 + S^2}}{T}\right\},$$

therefore

$$S + \sqrt{T^2 + S^2} = T\,e^{\frac{x}{T}}. \quad . \quad . \quad . \quad . \quad (B)$$

Again,

$$\{\sqrt{T^2 + S^2} - S\}\,\{\sqrt{T^2 + S^2} + S\} = T^2.$$

$$\therefore \quad \sqrt{T^2 + S^2} - S = \frac{T^2}{\sqrt{T^2 + S^2} + S} = \frac{T^2}{T\,e^{\frac{x}{T}}} = T\,e^{-\frac{x}{T}}. \quad (C)$$

Adding (B) and (C), we have

$$2\sqrt{T^2 + S^2} = T\left\{e^{\frac{x}{T}} + e^{-\frac{x}{T}}\right\};$$

therefore

$$\sqrt{T^2 + S^2} = \frac{T}{2}\left\{e^{\frac{x}{T}} + e^{-\frac{x}{T}}\right\};$$

therefore by (A), we get

$$y + T = \frac{T}{2}\left\{e^{\frac{x}{T}} + e^{-\frac{x}{T}}\right\},$$

If we move the axis of X down a distance equal to the length of string or chain whose weight is equal to T, the equation becomes

$$y = \frac{T}{2}\left\{e^{\frac{x}{T}} + e^{-\frac{x}{T}}\right\}.$$

(165.) **Find the area bounded by the catenary curve and the axis of x between the limits $x = 0$, and $x = x_1$.**

$$A = \int y\,dx = \frac{T}{2}\int_0^{x_1}\left(e^{\frac{x}{T}} + e^{-\frac{x}{T}}\right)dx,$$

$$\therefore\quad A = \frac{T^2}{2}\left[e^{\frac{x}{T}} - e^{-\frac{x}{T}}\right]_0^{x_1},$$

$$\therefore\quad A = \frac{T^2}{2}\left(e^{\frac{x_1}{T}} - e^{-\frac{x_1}{T}}\right).$$

Art. 49.—Areas in Terms of Polar Co-ordinates.

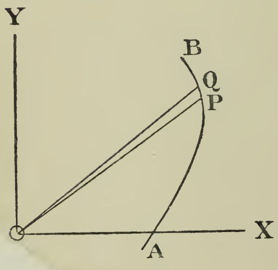


Fig. 21.

Let x, y, be the rectangular co-ordinates of the point P, on the curve A B. Then, if we ta[illegible] O X as the initial line, O, t[illegible] pole, the angle P O X = θ, t[illegible] vectorial angle, and O P the rad[illegible] vector ρ, of P, we have $x = \rho$ co[illegible], and $y = \rho \sin\theta$.

Let the point Q be indefinitely near to P, and denote the angle P O Q by $d\,\theta$, then an element of the curve is $\frac{1}{2}$ O P.O Q sin P O Q; but when P and Q are indefinitely near O P = O Q, and denoting the element of the area by d A, we have

$$d\,A = \tfrac{1}{2}\rho^2\,d\,\theta,$$

since sin $d\,\theta = d\,\theta$, when $d\,\theta$ is indefinitely small; therefore the area of any continuous curve in polar co-ordinates is given by the formula

$$A = \tfrac{1}{2}\int \rho^2\,d\,\theta. \quad . \quad . \quad . \quad . \quad \text{(D)}$$

(166.) **Find the Area of the Curve whose Equation is** $\rho = a\,(1 + \cos\theta)$, (*Cardioid*). It is obvious that the initial line divides it symmetrically, and that the whole area is

$$A = \int \rho^2\,d\,\theta;$$

that is,

$$A = a^2\int_0^\pi (1 + \cos\theta)^2\,d\,\theta;$$

therefore

$$A = a^2\left[\theta + 2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\,\theta}{4}\right]_0^\pi.$$

$$\therefore\quad A = \frac{3\,\pi\,a^2}{2}.$$

(167.) **Find the Area of the Lemniscate whose Equation is** $\rho^2 = a^2\cos 2\,\theta$. This curve, Fig. 22, has two loops, and its whole area is therefore

$$A = 4 \times \frac{1}{2}\int \rho^2\,d\,\theta.$$

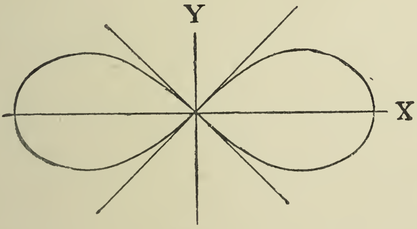


Fig. 22.

$$\therefore \quad A = 2\,a^2 \int^{\frac{\pi}{4}} \cos 2\,\theta\, d\,\theta = 2\,a^2 \left[\frac{\sin 2\,\theta}{2} \right]_0^{\frac{\pi}{4}};$$

that is,

$$A = 2\,a^2 \times \frac{1}{2} = a^2.$$

Examples.

1. Find the area of the curve whose equation is $\rho = a \cos \theta$. *Ans.* $\dfrac{\pi\, a^2}{4}$.

2. Find the area of a loop of the curve $\rho^2 = a^2 \cos 6\,\theta$. This curve has 6 loops. *Ans.* $\dfrac{a^2}{6}$.

3. Find the area bounded by the curve whose equation is $y = a\, e^{k\,x}$, and the axis of X between the limits $x = 0$, and $x = b$. *Ans.* $\dfrac{a}{k}\,(e^{k\,b} - 1)$.

4. Find the area contained by the curve $y = x^3 - 3\,x^2 + 4\,x$, and the axis of X. *Ans.* $\dfrac{x^4}{4} - x^3 + 2\,x^2$.

5. Find the area included between the curves $y^2 = 4\,m\,x$, $x^2 = 4\,m\,y$. *Ans.* $\dfrac{16\, m^2}{3}$.

6. Find the whole area of the ellipse whose equation is $a\,x^2 + 2\,h\,x\,y + b\,y^2 = 1$. *Ans.* $\dfrac{\pi}{\sqrt{a\,b - h^2}}$.

7. Find the area of the curve $m^2\,y^4 = x^4\,(m^2 - x^2)$ betw

the limits $x = m$ and $x = -\,m$. *Ans.* $\dfrac{8\,m^2}{5}$.

8. Find the whole area between the curve $x\,y^2 =$

$\times\,(2\,m - x)$, and the line $y = o$. *Ans.* $4\,\pi\,m^2$.

CHAPTER XVII.

AREAS OF SURFACES OF REVOLUTION.

ART. 50.—If a plane curve revolve about a fixed straight line in its plane, every point in the curve will, obviously, describe a circle, and the curve will generate a surface called a *surface of revolution.*

The line about which the curve revolves is called the *axis* of revolution.

The section of a surface of olution, by a plane perpen-ular to its axis, is a circle.

Let the curve P Q, Fig. , revolve about the axis of then we may assume that arc P Q is made up of an nite number of straight s of length $d\,s$.

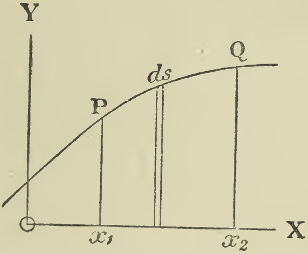


FIG. 23.

The surface generated by revolution of $d\,s$ about X is $2\,\pi\,y\,d\,s$, and denoting this element of surface by $d\,S$ e have

$$d\,S = 2\,\pi\,y\,d\,s,$$

erefore

$$S = 2\,\pi \int y\,d\,s + C.$$

Again,

$$d\,s = \sqrt{\left\{1 + \left(\frac{d\,y}{d\,x}\right)^2\right\}}\,d\,x,$$

therefore

$$S = 2\pi \int y \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}\, dx + C. \quad . \quad \text{(A)}$$

If $y = f(x)$ be the equation of the curve P Q, Fig. 23, then the surface generated by P Q is

$$S = 2\pi \int_{x_1}^{x_2} f(x) \sqrt{\left\{1 + f^1(x)^2\right\}}\, dx.$$

Similarly, we may show that the surface generated by the revolution of a plane curve about the axis of Y, is

$$S = 2\pi \int x \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}\, dx + C. \quad . \quad \text{(B)}$$

(168.) **Find the surface of a sphere.**

The surface of a sphere is generated by the revolution of a circle about a diameter.

Let the circle, whose centre is at the origin, revolve about the axis of X; then its equation is $x^2 + y^2 = r^2$.

$$\therefore \quad y = \sqrt{r^2 - x^2}.$$

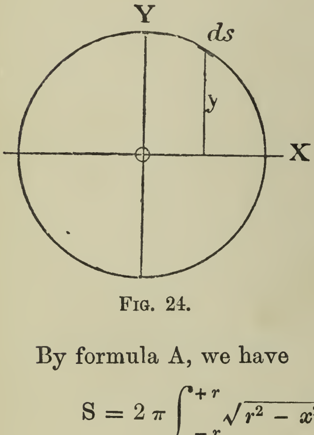


Fig. 24.

By formula A, we have

$$S = 2\pi \int_{-r}^{+r} \sqrt{r^2 - x^2} \sqrt{\left\{1 + \frac{x^2}{r^2 - x^2}\right\}}\, dx,$$

that is,

$$S = 2\pi \int_{-r}^{+r} r\, dx = 2\pi \Big[r x \Big]_{-r}^{+r}.$$

$$\therefore \quad S = 2\pi \{r^2 - (-r^2)\} = 4\pi r^2.$$

It will be noticed that the surface of a sphere is equal the curved surface of its circumscribing cylinder.

(169.) **Find the surface generated by the revolution of the parabola $y^2 = 4\,a\,x$ about the axis of X, between the limits $x = 0$ and $x = a$.**

$$S = 2\,\pi \int \sqrt{4\,a\,x}\ \sqrt{\left\{1 + \frac{a}{x}\right\}}\, d\,x + C \text{ by A.}$$

Therefore

$$S = 4\,\pi\,\sqrt{a} \int_0^a \sqrt{(x + a)}\; d\,x;$$

$2y\,dy = a\,x\,d$

$\frac{dy}{dx} = \frac{ax}{2y}$

that is,

$$S = 8\,\pi\,\sqrt{a}\left[\frac{(x + a)^{\frac{3}{2}}}{3}\right]_0^a.$$

$$\therefore\quad S = \frac{8\,\pi\,a^2}{3}\,(2^{\frac{3}{2}} - 1).$$

(170.) **Find the surface generated by the revolution of the catenary $y = \frac{c}{2}\left\{e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right\}$ about the axis of Y.**

By B, we have

$$S = 2\,\pi \int^x x \sqrt{\left\{1 + \left(\frac{d\,y}{d\,x}\right)^2\right\}}\, d\,x.$$

Now,

$$\sqrt{\left\{1 + \left(\frac{d\,y}{d\,x}\right)^2\right\}} = \frac{1}{2}\left\{e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right\},$$

therefore

$$S = \pi \int_0^x x \left\{e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right\} d\,x.$$

$$\int_0^x x \left\{e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right\} d\,x = c\,x\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right) - c^2\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right) + 2\,c^2.$$

$$\therefore\quad S = \pi\,c\left\{x\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right) - c\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right) + 2\,c.\right\}.$$

(171.) **Find the surface generated by the revolution of the curve $y = a e^{cx}$ round the axis of X.**

$$S = 2\pi a \int e^{cx}\sqrt{\{1 + (a c)^2 e^{2cx}\}}\, d x + C.$$

Let $a c e^{cx} = \tan\theta$, and the expression becomes

$$S = \frac{2\pi}{c^2}\int \frac{d\theta}{\cos^3\theta} = \frac{2\pi}{c^2}\left\{\frac{\sin\theta}{2\cos^2\theta} + \frac{1}{4}\log\left(\frac{1+\sin\theta}{1-\sin\theta}\right)\right\} + C.$$

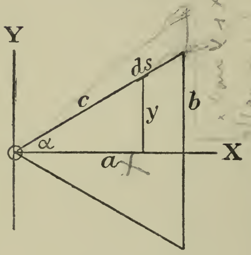


FIG. 25.

(172.) **Find the surface generated by the revolution of a right-angled triangle about the side a.**

The side c will generate the surface of a cone whose convex area is required. Denote the semi-vertical angle by α; therefore the required area

$$S = 2\pi \int y\, d s.$$

$$= 2\pi \tan\alpha \int_0^a x\sqrt{1 + \tan^2\alpha}\, d x,$$

since $y = x\tan\alpha$.

$$\therefore\quad S = 2\pi\tan\alpha\sqrt{1+\tan^2\alpha}\int_0^a x\, d x;$$

that is,

$$S = 2\pi\tan\alpha\sqrt{1+\tan^2\alpha} \times \frac{a^2}{2},$$

and since $\tan\alpha = \frac{b}{a}$, we have

$$S = \pi b\sqrt{a^2 + b^2};$$

that is, the area is the product of semi-circumference of base and slant side.

(173.) **Find the surface of a spherical ring.**

A spherical ring is generated by the revolution of a circle about a straight line in its plane. Let the radius of the ring be r, and let the distance of the centre of the circle from the straight line be R. Let $a\, b\, a_1\, b_1$, Fig. 26, be the generating circle, and let $a\, a_1$ D be drawn perpendicular to the axis of X, then an element of the area of the ring is equal to the sum of the elementary zones generated by the arcs $a\, b$ and $a_1\, b_1$; therefore

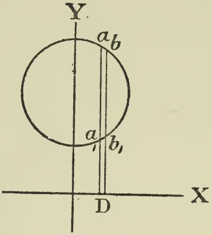


FIG. 26.

$$d\,\mathrm{S} = 2\,\pi\,(y_1 + y_2)\,d\,s$$

where

$$y_1 = a\,\mathrm{D} \quad \text{and} \quad y_2 = a_1\,\mathrm{D};$$

that is,

$$d\,\mathrm{S} = 4\,\pi\,\mathrm{R}\,d\,s, \text{ since } \mathrm{R} = \tfrac{1}{2}\,(y_1 + y_2),$$

because the circle is divided symmetrically by the straight line through its centre parallel to the axis of X; therefore

$$\mathrm{S} = 4\,\pi\,\mathrm{R}\int_0^{\frac{s}{2}} d\,s;$$

that is,

$$\mathrm{S} = 2\,\pi\,\mathrm{R}\,s$$

where s is the circumference of the generating circle; that is,

$$\mathrm{S} = 2\,\pi\,\mathrm{R} \times 2\,\pi\,r = 4\,\pi^2\,\mathrm{R}\,r.$$

ART. 51.—Guldinus' Theorems.

First. **The surface generated by the revolution of a curve about any external axis in its plane, is equal to the product of the perimeter of the curve into the distance through which the centre of gravity of the curve moves.**

Let $\bar{y}$ denote the distance of the centre of gravity of the perimeter of the curve from the axis, s the length of the curve; then we have, by the method of moments,

$$\bar{y}\,s = \int y\,d\,s;$$

therefore

$$2\pi \bar{y} s = 2\pi \int y \, ds.$$

This proves the theorem, since $2\pi\bar{y}$ is the dis through which the centre of gravity of the curve moves one revolution, and s is the length of the curve.

Second. **The volume generated by the revolution a plane area about an external axis in its plane equal to the product of th area into the distance throug which its centre of are moves.**

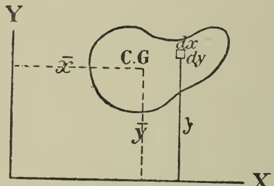


FIG. 27.

To prove this theorem; le the whole area be denoted by A and let O X denote the extern axis about which it revolves $\bar{x}$, $\bar{y}$, the co-ordinates of its centr of area; then an element of area $dA = dy\,dx$, and we hav the moment of the whole area, A, about O X, equal to th sum of the moments of its elements about O X; therefore

$$\bar{y} A = \int y \, dA;$$

that is,

$$\bar{y} A = \int\int y \, dx \, dy = \frac{1}{2}\int y^2 \, dx;$$

therefore

$$2\pi\bar{y} A = \pi \int y^2 \, dx.$$

This integral is taken the whole distance round the curve.

Now $2\pi\bar{y}$ is the distance through which the centre of gravity of the area moves in one revolution, and A is its whole area, which establishes the theorem, since $\pi\int y^2\,dx$ represents the whole volume.

EXAMPLES.

Find the surface generated by the revolution of the line $y = mx + c$ about the axis of X between the limits $x = 0$ and $x = a$.

Ans. $2\pi\sqrt{(1+m^2)}\left(\frac{m a^2}{2} + a c\right)$.

2. Find the surface generated by the revolution of the curve $y = a x^3$ about the axis of X.

Ans. $\frac{\pi}{27 a}\left(1 + 9 a^2 x^4\right)^{\frac{3}{2}} + C$.

3. Find the surface generated by the revolution of the cycloid,

$$x = a(\theta - \sin\theta)$$
$$y = a(1 - \cos\theta),$$

round its base.

Ans. $\frac{64\pi a^2}{3}$.

4. Find the surface generated by the revolution of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

about its minor axis.

Ans. $\pi\left\{2a^2 + \frac{a b^2}{\sqrt{a^2 - b^2}}\log\left(\frac{a + \sqrt{a^2 - b^2}}{a - \sqrt{a^2 - b^2}}\right)\right\}$.

5. Find the surface generated by the revolution of an arc of a circle of radius r, subtending an angle of 60° at the centre, about a line in its plane parallel to its chord and at a distance d from it.

Ans. $\frac{2\pi^2 r}{3}\left\{d + r\sqrt{3}\left(\frac{\sqrt{3}}{\pi} - \frac{1}{2}\right)\right\}$.

6. Find the surface generated by the revolution of a semicircular arc about a tangent at its middle point.

Ans. $2\pi r^2(\pi - 2)$.

CHAPTER XVIII.

VOLUMES OF SOLIDS OF REVOLUTION.

ART. 52.—If a plane curve makes one complete revolution about a line in its plane, it will describe what is called a *solid of revolution.* Taking this line as the axis of X, the area of the circle described by a point P on the curve is πy^2, and an element of the volume is the area of this circle multiplied by dx; therefore

$$dV = \pi y^2 \, dx,$$

where V is the volume.

$$\therefore \quad V = \pi \int y^2 \, dx + C,$$

where C is a constant.

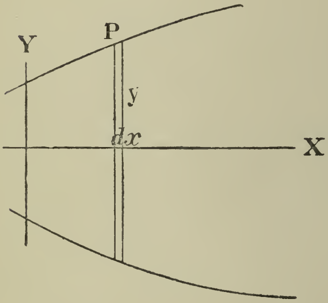


FIG. 28.

(174.) **Find the volume of a sphere. Taking the axes and origin as in Example 168,**

We have

$$V = \pi \int_{-r}^{+r} (r^2 - x^2) \, dx;$$

that is,

$$V = \pi \left[r^2 x - \frac{x^3}{3} \right]_{-r}^{+r}$$

$$= \pi \left\{ r^3 - \frac{r^3}{3} - \left(-r^3 + \frac{r^3}{3} \right) \right\}.$$

$$\therefore \quad V = \frac{4}{3} \pi r^3.$$

(175.) **Find the volume of a cone.**—Taking the cone mentioned in Example 172, we have

$$V = \pi \int_0^a x^2 \tan^2 \alpha \, d\,x = \pi \tan^2 \alpha \int_0^a x^2 \, d\,x$$

$$= \pi \tan^2 \alpha \, \frac{a^3}{3} = \pi \frac{b^2}{a^2} \times \frac{a^3}{3} = \pi b^2 \times \frac{a}{3},$$

that is, the volume of a cone is equal to the product of the area of its base and one-third its vertical height.

(176.) **Find the volume generated by the revolution of the parabola $y^2 = 4\,a\,x$ about its axis.**

Here

$$V = 4\,\pi \int_0^x a\,x\,d\,x;$$

therefore

$$V = 4\,\pi\,a \left[\frac{x^2}{2} \right]_0^x$$

$$= \frac{x}{2} \times 4\,\pi\,a\,x = \pi\,y^2 \times \frac{x}{2},$$

that is, the volume is equal to one-half the volume of the circumscribing cylinder.

(177.) **Find the volume generated by the revolution of the cycloid $x = a\,(\theta - \sin\theta)$, $y = a\,(1 - \cos\theta)$, about the axis of X.**

Here

$$d\,V = \pi\,a^3 (1 - \cos\theta)^3 \, d\,\theta;$$

therefore

$$V = \pi\,a^3 \int_0^{2\pi} (1 - \cos\theta)^3 \, d\,\theta,$$

that is,

$$V = \pi\,a^3 \int_0^{2\pi} (1 - 3\cos\theta + 3\cos^2\theta - \cos^3\theta)\, d\,\theta$$

$$= \pi\,a^3 \left[\theta - 3\sin\theta + \frac{3\,\theta}{2} + \frac{3\sin 2\,\theta}{4} - \sin\theta + \frac{\sin^3\theta}{3} \right]_0^{2\pi}.$$

$$\therefore \;\; V = 5\,\pi^2 a^3.$$

EXAMPLES.

1. Find the volume generated by the revol tan triangle about a line through its vertex parallel to

Ans. $\frac{2}{3}\pi h^2 b$, where h is its height and b its base.

2. Find the volume generated by the revolution of a s ment of a circle about its chord.

Ans. $2\pi r\left\{\frac{(2r^2+d^2)\cos\theta}{3}-rd\right\}$, where r is the radius, d the distance of the chord from the centre, and $\sin\theta=\frac{d}{r}$.

3. Find the volume generated by the revolution of the cycloid $x=a(\theta-\sin\theta)$, $y=a(1-\cos\theta)$, about the axis of Y.

Ans. $6\pi^3a^3$.

4. Find the volume of the part cut off from a sphere, c radius r, by a plane whose distance is $\frac{r}{3}$ from its centre.

Ans. $\frac{28}{81}\pi r^3$.

5. Find the volume of the ellipsoid whose equation is

$$\frac{x^2}{l^2}+\frac{y^2}{m^2}+\frac{z^2}{n^2}=1.$$

Ans. $\frac{4\pi lmn}{3}$.

6. Find the volume of the frustum of a sphere.

Ans. $\frac{\pi k}{6}\left\{k^2+3(a_1{}^2+a_2{}^2)\right\}$ where k is its height, an a_1 and a_2 the radii of its ends.

(17 Find the ratio of a hemisphere, paraboloid and cone, on cone m base and all of the same altitude.

4 : 3 : 2.

nd the volume generated by the revolution of the se $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about a line in its plane at a distance d tl its centre.

Ans. $2\pi^2 a b d$.

CHAPTER XIX.

LENGTHS OF CURVES.

ART. 53.—The process of determining the length of an arc of a curve is called its *rectification*.

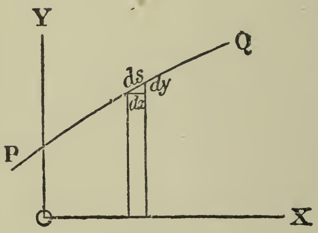


FIG. 29.

Let P Q, Fig. 29, be an arc of a curve whose length is required; then denoting an indefinitely small portion of it by $d s$, we have

$$(d s)^2 = (d x)^2 + (d y)^2;$$

therefore

$$d s = \sqrt{\left\{1 + \left(\frac{d y}{d x}\right)^2\right\}}\, d x,$$

or

$$d s = \sqrt{\left\{1 + \left(\frac{d x}{d y}\right)^2\right\}}\, d y;$$

therefore the arc

$$s = \int \sqrt{\left\{1 + \left(\frac{d y}{d x}\right)^2\right\}}\, d x = \int \sqrt{\left\{1 + \left(\frac{d x}{d y}\right)^2\right\}}\, d y. \quad \text{(A)}$$

(178.)—Find the length of the circumference of a circle.

Let the equation of the circle be

$$x^2 + y^2 = r^2.$$

By A, we have

$$s = \int \sqrt{\left(1 + \frac{x^2}{y^2}\right)}\, d x,$$

since

$$\frac{dy}{dx} = -\frac{x}{y}.$$

$$\therefore \quad s = \int \frac{r\,dx}{\sqrt{r^2 - x^2}},$$

and the whole circumference is equal to four times this integral between the limits $x = 0$ and $x = r$; therefore the whole circumference

$$s = 4\int_0^r \frac{r\,dx}{\sqrt{r^2 - x^2}}$$

$$= 4\,r\left[\sin^{-1}\left(\frac{x}{r}\right)\right]_0^r = 4\,r\,\frac{\pi}{2} = 2\,\pi\,r.$$

(179.) **Find the length of the arc of the parabola,** $y^2 = 4\,a\,x$.

Here

$$\frac{dy}{dx} = \frac{2\,a}{y},$$

and by formula A,

$$s = \int\sqrt{\left(1 + \frac{4\,a^2}{y^2}\right)}\,dx = \int\sqrt{\left(1 + \frac{a}{x}\right)}\,dx. \qquad (a)$$

Let

$$x = z^2 \qquad \therefore \quad dx = 2\,z\,dz,$$

and (a) becomes

$$s = 2\int\sqrt{(z^2 + a)}\,dz$$

$$= 2\left\{\frac{z\sqrt{(z^2 + a)}}{2} + \frac{a}{2}\log\left(z + \sqrt{z^2 + a}\right)\right\} + C.$$

$$\therefore \quad s = \sqrt{x}\,\sqrt{x + a} + a\log\left(\sqrt{x} + \sqrt{x + a}\right) + C.$$

Suppose the length of the arc, from the vertex to the end of the latus-rectum, is required, we must integrate between the limits $x = 0$ and $x = a$.

$$\therefore \quad s = \sqrt{a}\,\sqrt{a + a} + a\log\left(\sqrt{a} + \sqrt{a + a}\right) - a\log\sqrt{a},$$

that is,

$$s = a\sqrt{2} + a \log (1 + \sqrt{2}).$$

(180.) **Find the length of an arc of the catenary**

$$y = \frac{a}{2}\left\{e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right\}.$$

Here

$$\frac{dy}{dx} = \frac{1}{2}\left\{e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right\}.$$

therefore

$$s = \int\sqrt{\left[1 + \frac{1}{4}\left\{e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right\}^2\right]}\,dx = \frac{1}{2}\int\left\{e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right\}dx,$$

$$\therefore\quad s = \frac{a}{2}\left\{e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right\}.$$

The length of the arc from the vertex to the point $x = a$ is therefore

$$s = \frac{a}{2}(e - e^{-1}).$$

(181.) **Find the length of the cycloid**

$$x = a(\theta - \sin\theta)$$
$$y = a(1 - \cos\theta).$$

Here

$$\frac{dy}{dx} = \frac{\sin\theta}{1 - \cos\theta}$$

and

$$dx = a(1 - \cos\theta)\,d\theta.$$

Substituting in (A) we have

$$s = a\int\sqrt{\left\{1 + \frac{\sin^2\theta}{(1 - \cos\theta)^2}\right\}}\,(1 - \cos\theta)\,d\theta,$$

that is,

$$s = a\sqrt{2}\int\sqrt{1 - \cos\theta}\,d\theta = 2a\int\sin\tfrac{1}{2}\theta\,d\theta.$$

$$\therefore\quad s = 2a\Big[-2\cos\tfrac{1}{2}\theta\Big]_0^{2\pi}$$

for the whole length of the curve; therefore

$$s = 8\,a.$$

That is, four times the diameter of the generating circle.

EXAMPLES.

1. Find the length of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = r^{\frac{2}{3}}$.

Ans. $s = \dfrac{3}{2}\, r^{\frac{1}{3}}\, x^{\frac{2}{3}} + c.$

2. Find the length of the curve whose equation is $a\,y^2 = x^3$.

Ans. $s = \dfrac{1}{\sqrt{a}} \left\{ \dfrac{4\,a}{9} + x \right\}^{\frac{3}{2}} + c.$

3. Find the length of the arc of the logarithmic curve $y = e^x$.

Ans. $s = \log \dfrac{y}{1 + \sqrt{1 + y^2}} + \sqrt{1 + y^2} + C.$

4. The chain of a suspension bridge hangs in a parabolic arc between two supports of the same height, the distance between the supports being 70 feet, and the dip being 35 feet, find the length of the chain between the points of support.

Ans. 103·5 feet nearly.

CHAPTER XX.

FOURIER'S EXPANSION.

Art. 54.—If $f(t)$ be any periodic function of the time t, the periodic time being T, then

$$f(t) = f(t + n\mathrm{T})$$

where n is an integer.

Any periodic curve may be represented by a constant term A_0 say,

$+$ a sine curve of the same T as the original curve.
$+$ a cosine ” ” ”
$+$ a sine curve of periodic time $\frac{\mathrm{T}}{2}$.
$+$ a cosine ” ” ”
$+$ a sine ” ” $\frac{\mathrm{T}}{3}$.
$+$ a cosine ” ” ”
$+$ a sine ” ” $\frac{\mathrm{T}}{4}$.
$+$ a cosine ” ” ”

. . . . .

$+$ a sine ” ” $\frac{\mathrm{T}}{n}$.
$+$ a cosine ” ” ”

. . . . .

Therefore

$$
\begin{aligned}
f(t) = A_0 + A_1 \sin \frac{2\pi t}{T} + B_1 \cos \frac{2\pi t}{T} \\
+ A_2 \sin \frac{4\pi t}{T} + B_2 \cos \frac{4\pi t}{T} \\
+ A_3 \sin \frac{6\pi t}{T} + B_3 \cos \frac{6\pi t}{T} \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
+ A_n \sin \frac{2n\pi t}{T} + B_n \cos \frac{2n\pi t}{T}, \text{ \&c., \&c.} \quad . \quad (1)
\end{aligned}
$$

where A_1, B_1, A_2, B_2, &c., are constants.

We shall show how the values of these constants may be found when the curve representing $f(t)$ is given.

Multiply (1) by dt, and integrate between the limits 0 and T thus:

$$
\begin{aligned}
\int_0^T f(t)\,dt = A_0 \int_0^T dt + A_1 \int_0^T \sin \frac{2\pi t}{T}\,dt \\
+ B_1 \int_0^T \cos \frac{2\pi t}{T}\,dt + \text{\&c.}
\end{aligned}
$$

$$
\therefore \int_0^T f(t)\,dt = A_0 T + 0 + 0, \text{ \&c.}
$$

All the terms on the right vanish except the first.

Suppose we take the term

$$
A_n \int_0^T \sin \frac{2n\pi t}{T}\,dt,
$$

which is

$$
= A_n \left[-\frac{T}{2n\pi} \cos \frac{2n\pi t}{T} \right]_0^T = -A_n \frac{T}{2n\pi}(1-1) = 0.
$$

In like manner we may show that all the terms, but the first, vanish; therefore

$$
A_0 = \frac{1}{T} \int_0^T f(t)\,dt,
$$

that is, A_0 is the average height of the original curve.

To get A_1, multiply all along by $\sin \frac{2\pi t}{T}\,dt$ and integrate between the same limits as before. Thus

$$\int_0^T f(t)\sin\frac{2\pi t}{T}\,dt = A_0\int_0^T \sin\frac{2\pi t}{T}\,dt$$

$$+ A_1\int_0^T \sin^2\frac{2\pi t}{T}\,dt + \&c.$$

$$\therefore \quad \int_0^T f(t)\sin\frac{2\pi t}{T}\,dt = 0 + A_1\frac{T}{2} + 0, \&c.$$

All the terms on the right vanish except the second.

We have shown that the first term will vanish in finding A_0. Suppose we take the term

$$A_n\int_0^T \sin\frac{2n\pi t}{T}\sin\frac{2\pi t}{T}\,dt,$$

which is

$$\frac{A_n}{2}\int_0^T\left\{\cos\frac{2\pi(n-1)t}{T} - \cos\frac{2\pi(n+1)t}{T}\right\}dt$$

$$= \frac{A_n}{2}\left[\frac{T}{2\pi(n-1)}\sin\frac{2\pi(n-1)t}{T} - \frac{T}{2\pi(n+1)}\sin\frac{2\pi(n+1)t}{T}\right.$$

$= 0$ obviously; therefore

$$A_1 = \frac{2}{T}\int_0^T f(t)\sin\frac{2\pi t}{T}\,dt.$$

Similarly,

$$B_1 = \frac{2}{T}\int_0^T f(t)\cos\frac{2\pi t}{T}\,dt.$$

$$A_2 = \frac{2}{T}\int_0^T f(t)\sin\frac{4\pi t}{T}\,dt.$$

$$B_2 = \frac{2}{T}\int_0^T f(t)\cos\frac{4\pi t}{T}\,dt.$$

. . . . .

$$A_n = \frac{2}{T}\int_0^T f(t)\sin\frac{2n\pi t}{T}\,dt.$$

$$B_n = \frac{2}{T}\int_0^T f(t)\cos\frac{2n\pi t}{T}\,dt.$$

The following examples will show how these integrals are worked out.

(182.) Develop the periodic curve

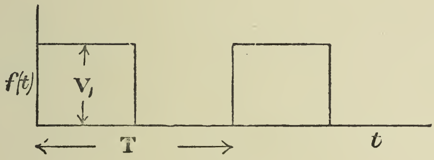


FIG. 30.

Here we have

$$f(t) = A_0 + A_1\sin\frac{2\pi t}{T} + B_1\cos\frac{2\pi t}{T} + A_2\sin\frac{4\pi t}{T}$$

$$+ B_2\cos\frac{4\pi t}{T} + \ldots \&c.$$

Multiply by dt and integrate between the limits 0 and T, and we have

$$A_0 T = \int_0^T f(t)\,dt.$$

$$\therefore\quad A_0 = \frac{1}{T}\int_0^T f(t)\,dt = \text{average height of } f(t) = \frac{V_1}{2}.$$

To get A_1, multiply by $\sin\frac{2\pi t}{T}\,dt$, and integrate between the limits 0 and T, thus:

$$\int_0^T f(t)\sin\frac{2\pi t}{T}\,dt = A_1\int_0^T \sin^2\left(\frac{2\pi t}{T}\right)dt = A_1\frac{T}{2}.$$

$$\therefore\quad A_1 = \frac{2}{T}\int_0^T f(t)\sin\frac{2\pi t}{T}\,dt = \frac{2}{T}\int_0^{\frac{T}{2}} V_1\sin\frac{2\pi t}{T}\,dt$$

$$+\frac{2}{T}\int_{\frac{T}{2}}^{T} 0 \times \sin\frac{2\pi t}{T}\,dt,$$

that is,

$$A_1 = \frac{2V_1}{T}\left[-\frac{T}{2\pi}\cos\frac{2\pi t}{T}\right]_0^{\frac{T}{2}}.$$

$$\therefore\quad A_1 = \frac{2V_1}{\pi}.$$

Similarly, by multiplying by $\cos\frac{2\pi t}{T}\,dt$, and integrating between the limits 0 and T, we get

$$B_1 = \frac{2V_1}{T}\left[\frac{T}{2\pi}\sin\frac{2\pi t}{T}\right]_0^{\frac{T}{2}}.$$

$$\therefore\quad B_1 = 0.$$

To get A_2, multiply by $\sin\frac{4\pi t}{T}\,dt$ and integrate, we get

$$A_2 = \frac{2V_1}{T}\int_0^{\frac{T}{2}}\sin\frac{4\pi t}{T}\,dt = \frac{2V_1}{T}\left[-\frac{T}{4\pi}\cos\frac{4\pi t}{T}\right]_0^{\frac{T}{2}}.$$

$$\therefore\quad A_2 = 0.$$

In like manner we find that

$$B_2 = 0,\quad A_3 = \frac{2V_1}{3\pi},\quad B_3 = 0,\quad A_4 = 0,\quad B_4 = 0,$$

$$\text{and}\ \ A_5 = \frac{2V_1}{5\pi},\ \ \&c.$$

Therefore

$$f(t) = \frac{V_1}{2} + \frac{2V_1}{\pi}\left\{\sin\frac{2\pi t}{T} + \frac{1}{3}\sin\frac{6\pi t}{T} + \frac{1}{5}\sin\frac{10\pi t}{T}\right.$$

$$\left.\ldots\ldots + \frac{1}{n}\sin\frac{2\pi n t}{T} + \&c.\right\},$$

where n is an odd integer.

(183.) Develop the periodic function

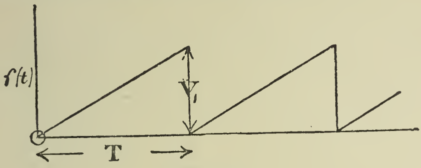


FIG. 31.

Here $A_0 = \frac{1}{T}\int_0^T f(t)\,dt$, and $f(t) = Kt$ where K is a constant, and $f(t) = V_1$ when $t = T$; therefore $V_1 = KT$, that is, $K = \frac{V_1}{T}$.

Hence $f(t) = \frac{V_1}{T}t$. Therefore

$$A_0 = \frac{V_1}{T^2}\int_0^T t\,dt = \frac{V_1}{T^2}\left[\frac{t^2}{2}\right]_0^T = \frac{V_1}{2}.$$

Again,

$$A_1 = \frac{2}{T}\int_0^T \frac{V_1 t}{T}\sin\frac{2\pi t}{T}\,dt = \frac{2V_1}{T^2}\int_0^T t\sin\frac{2\pi t}{T}\,dt,$$

d on integrating this by parts, we get

$$A_1 = \frac{2V_1}{T^2}\left[-\frac{T}{2\pi}t\cos\frac{2\pi t}{T} + \left(\frac{T}{2\pi}\right)^2\sin\frac{2\pi t}{T}\right]_0^T.$$

$$\therefore\ A_1 = -\frac{V_1}{\pi}.$$

Again,

$$B_1 = \frac{2V_1}{T^2}\int_0^T t\cos\frac{2\pi t}{T}\,dt,$$

nd on integrating by parts we have

$$B_1 = \frac{2V_1}{T^2}\left[\frac{Tt}{2\pi}\sin\frac{2\pi t}{T} + \left(\frac{T}{2\pi}\right)^2\cos\frac{2\pi t}{T}\right]_0^T.$$

$$\therefore\ B_1 = 0.$$

Similarly, we can show that B_2, B_3, &c., all vanish. Again,

$$A_2 = \frac{2\,V_1}{T^2}\int_0^T t \sin\frac{4\pi t}{T}\,dt$$

$$= \frac{2\,V_1}{T^2}\left[-\frac{T\,t}{4\pi}\cos\frac{4\pi t}{T} + \left(\frac{T}{4\pi}\right)^2 \sin\frac{4\pi t}{T}\right]_0^T.$$

$$\therefore\quad A_2 = -\frac{V_1}{2\pi}.$$

$$A_3 = \frac{2\,V_1}{T^2}\int_0^T t \sin\frac{6\pi t}{T}\,dt$$

$$= \frac{2\,V_1}{T^2}\left[-\frac{T\,t}{6\pi}\cos\frac{6\pi t}{T} + \left(\frac{T}{6\pi}\right)^2 \sin\frac{6\pi t}{T}\right]_0^T.$$

$$\therefore\quad A_3 = -\frac{V_1}{3\pi}.$$

Similarly,

$$A_4 = -\frac{V_1}{4\pi},\ \text{\&c.}$$

Therefore

$$f(t) = \frac{V_1}{\pi}\left\{\frac{\pi}{2} - \sin\frac{2\pi t}{T} - \frac{1}{2}\sin\frac{4\pi t}{T} - \frac{1}{3}\sin\frac{6\pi t}{T} - \frac{1}{4}\sin\frac{8\pi t}{T} - \text{\&c.}\right\}.$$

(184.) Develop the periodic function, Fig. 32.

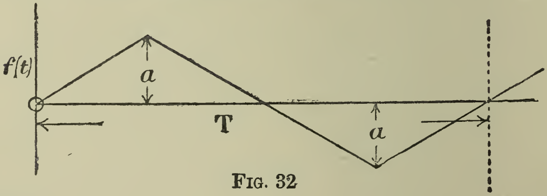


FIG. 32

Here

$$A_0 = \frac{1}{T}\int_0^T f(t)\,dt$$

= average height of $f(t)$, which is zero; therefore

$$A_0 = 0.$$

Again,

$$A_1 = \frac{2}{T}\int f(t)\sin\frac{2\pi t}{T}\,dt$$

Now, $f(t) = Kt$ between the limits $-\frac{T}{4}$ and $+\frac{T}{4}$; and when $t = \frac{T}{4}$, $f(t) = a$.

$$\therefore \quad a = K\frac{T}{4}.$$

$$\therefore \quad K = \frac{4a}{T}.$$

Again, we have $f(t) = 2a\left(1 - \frac{2t}{T}\right)$ between the limits $t = \frac{T}{4}$ and $t = \frac{3T}{4}$; therefore

$$A_1 = \frac{2}{T}\int_{-\frac{T}{4}}^{+\frac{T}{4}} \frac{4at}{T}\sin\frac{2\pi t}{T}\,dt + \frac{2}{T}\int_{\frac{T}{4}}^{\frac{3T}{4}} 2a\left(1 - \frac{2t}{T}\right)\sin\frac{2\pi t}{T}\,dt$$

$$= \frac{8a}{T^2}\int_{-\frac{T}{4}}^{+\frac{T}{4}} t\sin\frac{2\pi t}{T}\,dt + \frac{4a}{T}\int_{\frac{T}{4}}^{+\frac{3T}{4}} \sin\frac{2\pi t}{T}\,dt$$

$$- \frac{8a}{T^2}\int_{\frac{T}{4}}^{+\frac{3T}{4}} t\sin\frac{2\pi t}{T}\,dt.$$

$$\therefore \quad A_1 = \frac{8a}{T^2}\left[-\frac{Tt}{2\pi}\cos\frac{2\pi t}{T} + \left(\frac{T}{2\pi}\right)^2\sin\frac{2\pi t}{T}\right]_{-\frac{T}{4}}^{\frac{T}{4}}$$

$$+ \frac{4a}{T}\left[-\frac{T}{2\pi}\cos\frac{2\pi t}{T}\right]_{\frac{T}{4}}^{\frac{3T}{4}}$$

$$- \frac{8a}{T^2}\left[-\frac{Tt}{2\pi}\cos\frac{2\pi t}{T} + \left(\frac{T}{2\pi}\right)^2\sin\frac{2\pi t}{T}\right]_{\frac{T}{4}}^{\frac{3T}{4}}.$$

$$\therefore \quad A_1 = \frac{8a}{\pi^2}.$$

Similarly, we find that

$$B_1 = 0, \quad B_2 = 0, \quad B_3 = 0, \text{ \&c.,}$$
$$A_2 = 0, \quad A_4 = 0, \quad A_6 = 0, \text{ \&c.,}$$

and

$$A_3 = -\frac{8a}{9\pi^2}, \quad A_5 = \frac{8a}{25\pi^2}, \quad A_7 = -\frac{8a}{49\pi^2}, \text{ \&c.}$$

$$\therefore \; f(t) = \frac{8a}{\pi^2}\left\{\sin\frac{2\pi t}{T} - \frac{1}{9}\sin\frac{6\pi t}{T} + \frac{1}{25}\sin\frac{10\pi t}{T} - \frac{1}{49}\sin\frac{14\pi t}{T} + \text{\&c.}\right\}.$$

EXAMPLES.

1. Develop

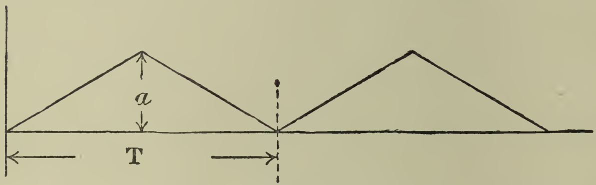


FIG. 33.

2. Develop

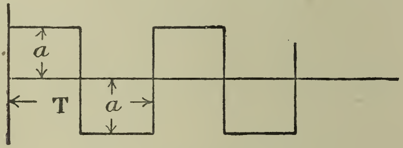


FIG. 34.

3. Develop

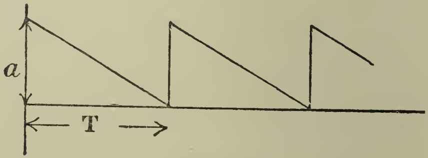


FIG. 35.

Development of

1. $$f(t) = \frac{a}{2} - \frac{4a}{\pi^2}\left\{\cos\frac{2\pi t}{T} + \frac{1}{9}\cos\frac{6\pi t}{T} + \frac{1}{25}\cos\frac{10\pi t}{T} + \frac{1}{49}\cos\frac{14\pi t}{T} + \&c.\right\}.$$

2. $$f(t) = \frac{4a}{\pi}\left\{\sin\frac{2\pi t}{T} + \frac{1}{3}\sin\frac{6\pi t}{T} + \frac{1}{5}\sin\frac{10\pi t}{T} + \&c.\right\}.$$

3. $$f(t) = \frac{a}{2} + \frac{a}{\pi}\left\{\sin\frac{2\pi t}{T} + \frac{1}{2}\sin\frac{4\pi t}{T} + \frac{1}{3}\sin\frac{6\pi t}{T} + \&c.\right\}.$$

CHAPTER XXI.

DIFFERENTIAL EQUATIONS.

Art. 55.—**A differential equation** is one which involves differential coefficients. It may or may not contain the primitive variables from which the differential coefficients are derived.

There are two classes of differential equations.

1st. *Differential equations in which all the differential coefficients involved have reference to a single independent variable.*

For example:

$$\frac{d^2 y}{d x^2} + a \frac{d y}{d x} + b y = 0$$

is a differential equation in which x is the independent and y the dependent variable.

2nd. *Differential equations involving partial differential coefficients which indicate the presence of two or more independent variables.*

For example:

$$d u = \left(\frac{d u}{d x}\right) d x + \left(\frac{d u}{d y}\right) d y$$

is a partial differential equation, having u for its dependent and x and y for its independent variables. A differential equation is said to be of the first, second, third . . . or nth order, according as it contains the first, second, third . . or nth differential coefficient.

$$\frac{d y}{d x} + \mathrm{A} y = \mathrm{B}$$

is of the first order.

$$\frac{d^2 x}{d t^2} + a \frac{d x}{d t} + b x = 0$$

is of the second order.

$$\frac{d^n y}{d x^n} + a \frac{d^{n-1} y}{d x^{n-1}} \cdot \cdot \cdot \cdot \; n y = 0$$

s of the *n*th order.

A differential equation is said to be of the first, second, third . . . or *n*th degree, according as it contains differential coefficients of the first, second, third . . . or *n*th degree.

$$\frac{d y}{d x} + a y = b$$

s of the first degree.

$$\left(\frac{d y}{d x}\right)^2 + a \frac{d y}{d x} + b y = 0$$

s of the second degree.

$$\left(\frac{d y}{d x}\right)^n + a \left(\frac{d y}{d x}\right)^{n-1} \cdot \cdot \cdot \cdot \; n y = 0$$

s of the *n*th degree.

ART. 56.—A differential equation is said to be linear if it admits of being expressed in the form

$$\frac{d^n y}{d x^n} + a \frac{d^{n-1} y}{d x^{n-1}} + b \frac{d^{n-2} y}{d x^{n-2}} + \; . \; . \; . \; p y = q,$$

in which a, b, . . . p, and q are either constants or functions of the independent variable, x, only.

If a, b, . . . p be constants, it is said to be a linear differential equation with constant coefficients; for example:

$$\frac{d^4 y}{d x^4} + 4 \frac{d^3 y}{d x^3} + 6 \frac{d^2 y}{d x^2} + 4 \frac{d y}{d x} + y = 0$$

is a linear differential equation with constant coefficients.

If a, b, . . . p are functions of x, it is said to be a differential equation with variable coefficients; for example:

$$(1 - x^2) \frac{d^2 y}{d x^2} - x \frac{d y}{d x} - 2 y = 0$$

is a differential equation with variable coefficients.

ART. 57.—**Solution of differential equations of the first order and degree between two variables.**

The general form of differential equations of the first order and degree is

$$\text{A}\, d x + \text{B}\, d y = 0,$$

A and B being functions of the variables x and y, in which form either x or y may be regarded as the independent variable.

The above equation can always be solved if the variables A and B admit of being separated; that is, if the equation admits of being expressed in the form

$$\text{X}\, d x + \text{Y}\, d y = 0,$$

in which X and Y are functions of x and y respectively. The solution will be

$$\int \text{X}\, d x + \int \text{Y}\, d y = k,$$

where k is a constant.

(185.) **To solve**

$$(x^2 - 3 x)\, d x + (y^2 - 2 y)\, d y = 0,$$

we have

$$\int (x^2 - 3 x)\, d x + \int (y^2 - 2 y)\, d y = k,$$

that is,

$$\frac{x^3}{3} - \frac{3 x^2}{2} + \frac{y^3}{3} - y^2 = k.$$

(186.) **To solve**

$$\sqrt{(a^2 + y^2)}\, d x + \sqrt{b^2 + x^2}\, d y = 0.$$

Here the variables may be separated by dividing by $\sqrt{(a^2 + y^2)(b^2 + x^2)}$; therefore we have

$$\frac{dx}{\sqrt{b^2 + x^2}} + \frac{dy}{\sqrt{a^2 + y^2}} = 0.$$

Therefore

$$\log (x + \sqrt{b^2 + x^2}) + \log (y + \sqrt{a^2 + y^2}) = k.$$

$$\therefore \quad \log (x + \sqrt{b^2 + x^2})(y + \sqrt{a^2 + y^2}) = k.$$

$$\therefore \quad (x + \sqrt{b^2 + x^2})(y + \sqrt{a^2 + y^2}) = k',$$

where $k' = e^k$.

Art. 58.—**Homogeneous differential equations of the form** $A\,dx + B\,dy = 0$ may be solved by assuming $y = vx$, and the variables x and v will then admit of being separated.

(187.) **To solve** $\{x + \sqrt{y^2 + x^2}\}\,dy - y\,dx = 0$.

Here let $y = vx$; therefore $dy = v\,dx + x\,dv$; therefore

$$\{x + x\sqrt{v^2 + 1}\}(v\,dx + x\,dv) - vx\,dx = 0.$$

On dividing by x

$$(1 + \sqrt{v^2 + 1})(v\,dx + x\,dv) - v\,dx = 0.$$

$$\therefore \quad v\sqrt{v^2 + 1}\,dx + x(1 + \sqrt{v^2 + 1})\,dv = 0.$$

On dividing by $vx\sqrt{v^2 + 1}$, we have

$$\frac{dx}{x} + \frac{dv}{v} + \frac{dv}{v\sqrt{v^2 + 1}} = 0.$$

Integrating we have

$$\log x + \log v + \log \frac{v}{1 + \sqrt{v^2 + 1}} = k,$$

that is,

$$\log \frac{v^2 x}{1 + \sqrt{v^2 + 1}} = k.$$

$$\therefore \quad \frac{y^2}{x + \sqrt{x^2 + y^2}} = e^k = K \text{ say}.$$

$$\therefore \quad y^2 = 2Kx + K^2.$$

K being an arbitrary constant.

ART. 59.—**Equations of the form**

$$(a x + b y + c)\, d x + (a' x + b' y + c')\, d y = 0$$

may generally be solved by assuming $x = x' - l$, a
$y = y' - m$, and the transformed equation becomes

$$(a x' + b y' - a l - b m + c)\, d x' + (a' x' + b' y' - a' l - b' m + c')\, d y' = 0.$$

Now let

$$a l + b m = c,$$
$$a' l + b' m = c',$$

and these equations will determine l and m, and the tran formed equation will become

$$(a x' + b y')\, d x' + (a' x' + b' y')\, d y' = 0.$$

This equation is integrable by the preceding example assuming $y' = v x'$, since it is rendered homogeneous.

(188.) **To solve the equation**

$$(3 x + y + 7)\, d x + (2 x + 5 y + 9)\, d y = 0.$$

Assuming $x = x' - l$, and $y = y' - m$, the equation becomes

$$(3 x' + y' - 3 l - m + 7)\, d x' + (2 x' + 5 y' - 2 l - 5 m + 9)\, d y' = 0.$$

Let

$$3 l + m = 7,$$
$$2 l + 5 m = 9,$$

therefore

$$l = 2, \qquad m = 1,$$

and the equation becomes

$$(3 x' + y')\, d x' + (2 x' + 5 y')\, d y' = 0.$$

which is homogeneous.

Assume $y' = v x'$ and this equation becomes

$$(3 + 3 v + 5 v^2)\, d x' + x' (2 + 5 v)\, d v = 0.$$

$$\therefore \quad \frac{d x'}{x'} + \frac{(2 + 5 v)\, d v}{3 + 3 v + 5 v^2} = 0.$$

$$\therefore \quad \log x' + \int \frac{(2 + 5 v)\, d v}{3 + 3 v + 5 v^2} = k.$$

$$\therefore \quad \log x' + \frac{1}{2} \log (3 + 3v + 5v^2)$$

$$+ \frac{1}{\sqrt{51}} \tan^{-1} \frac{(10v + 3)}{\sqrt{51}} = k,$$

by Example 133. On substituting in terms of x and y we have

$$\log (x + 2) + \frac{1}{2} \log \left\{3 + 3\left(\frac{y+1}{x+2}\right) + 5\left(\frac{y+1}{x+2}\right)^2\right\}$$

$$+ \frac{1}{\sqrt{51}} \tan^{-1} \frac{(10y + 3x + 16)}{(x + 2)\sqrt{51}} = k.$$

ART. 60.—The solution of the equation $\frac{dy}{dx} + Ay = B$, in which A and B are functions of x, is of considerable importance in the working out of a great many electrical problems.

First suppose $B = 0$, and we have

$$\frac{dy}{dx} + Ay = 0.$$

$$\therefore \quad \frac{dy}{y} = -A\,dx,$$

and on integrating we have

$$\log y = -\int A\,dx + k,$$

where k is an arbitrary constant; therefore

$$y = e^{-\int A\,dx + k} = e^k e^{-\int A\,dx},$$

$$\therefore \quad y = K e^{-\int A\,dx}, \quad . \quad . \quad . \quad . \quad (a)$$

where $K = e^k$.

Again, assume K a function of x, $f(x)$,

$$\therefore \quad y = f(x)\, e^{-\int A\,dx}. \quad . \quad . \quad . \quad (\beta)$$

$$\therefore \quad \frac{dy}{dx} = -\mathrm{A} f(x)\, e^{-\int \mathrm{A}\,dx} + f'(x)\, e^{-\int \mathrm{A}\,dx}.$$

$$\therefore \quad \frac{dy}{dx} + \mathrm{A} f(x)\, e^{-\int \mathrm{A}\,dx} = f'(x)\, e^{-\int \mathrm{A}\,dx};$$

that is,

$$\frac{dy}{dx} + \mathrm{A}\,y = f'(x)\, e^{-\int \mathrm{A}\,dx} \text{ by } (\beta),$$

and on referring to the original equation we see that

$$f'(x)\, e^{-\int \mathrm{A}\,dx} = \mathrm{B},$$

therefore

$$f'(x) = e^{\int \mathrm{A}\,dx}\, \mathrm{B},$$

$$\therefore \quad f(x) = \int e^{\int \mathrm{A}\,dx}\, \mathrm{B}\, dx + k.$$

Substitute this value of $f(x)$ in equation (β) and we have

$$y = e^{-\int \mathrm{A}\,dx} \left\{ \int e^{\int \mathrm{A}\,dx}\, \mathrm{B}\, dx + k \right\} \quad . \quad . \quad (\gamma)$$

(189.) **Solve**

$$\frac{dy}{dx} + \frac{xy}{1+x^2} = \frac{1}{2x(1+x^2)}.$$

Here

$$\mathrm{A} = \frac{x}{1+x^2}. \qquad \therefore \quad \int \mathrm{A}\,dx = \frac{1}{2} \log (1+x^2).$$

$$\therefore \quad e^{-\int \mathrm{A}\,dx} = e^{-\frac{1}{2}\log(1+x^2)} = (1+x^2)^{-\frac{1}{2}}.$$

Also

$$e^{\int \mathrm{A}\,dx} = (1+x^2)^{\frac{1}{2}}.$$

Again,

$$\mathrm{B} = \frac{1}{2x(1+x^2)};$$

$$\therefore \quad \int e^{\int \mathrm{A}\,d x}\,\mathrm{B}\,d x = \int \frac{(1+x^2)^{\frac{1}{2}}\,d x}{2\,x\,(1+x^2)} = \int \frac{d x}{2\,x\,(1+x^2)^{\frac{1}{2}}}.$$

$$\therefore \quad \int e^{\int \mathrm{A}\,d x}\,\mathrm{B}\,d x = \frac{1}{2}\log\left(\frac{\sqrt{1+x^2}-1}{x}\right).$$

Substituting in (γ), we have

$$y = (1+x^2)^{-\frac{1}{2}}\left\{\frac{1}{2}\log\left(\frac{\sqrt{1+x^2}-1}{x}\right)+k\right\}.$$

(190.) **Solve**

$$\frac{d y}{d x} - \frac{3\,y}{x+a} = (x+a)^4.$$

Here

$$\mathrm{A} = -\frac{3}{x+a}, \quad \mathrm{B} = (x+a)^4.$$

Hence

$$\int \mathrm{A}\,d x = -\,3\log(x+a).$$

$$\therefore \quad e^{-\int \mathrm{A}\,d x} = e^{3\log(x+a)} = (x+a)^3,$$

and

$$e^{\int \mathrm{A}\,d x} = (x+a)^{-3}.$$

Also

$$\int e^{\int \mathrm{A}\,d x}\,\mathrm{B}\,d x = \int (x+a)^{-3}\,(x+a)^4\,d x = \frac{(x+a)^2}{2};$$

therefore

$$y = (x+a)^3\left\{\frac{(x+a)^2}{2}+k\right\}, \text{ by equation } (\gamma).$$

(191.) Given $\mathrm{V} = \mathrm{R}\,\mathrm{C} + \mathrm{L}\,\dfrac{d\,\mathrm{C}}{d\,t}$, find C where $\mathrm{V} = \sin n\,t$, R and L being constants. Here

$$\frac{d\,\mathrm{C}}{d\,t} + \frac{\mathrm{R}}{\mathrm{L}}\,\mathrm{C} = \frac{1}{\mathrm{L}}\sin n\,t.$$

In this example we have C, t, $\frac{\mathrm{R}}{\mathrm{L}}$ and $\frac{1}{\mathrm{L}}$ sin $n\,t$ instead of y, x, A and B respectively in equation (γ).

$$\int \mathrm{A}\, d\,x = \int \frac{\mathrm{R}}{\mathrm{L}}\, d\,t = \frac{\mathrm{R}\,t}{\mathrm{L}}.$$

$$\therefore \quad e^{-\int \mathrm{A}\, d\,x} = e^{-\frac{\mathrm{R}\,t}{\mathrm{L}}}.$$

$$\int e^{\int \mathrm{A}\, dx}\, \mathrm{B}\, d\,x = \frac{1}{\mathrm{L}} \int e^{\frac{\mathrm{R}\,t}{\mathrm{L}}} \sin n\,t\, d\,t$$

$$= \frac{e^{\frac{\mathrm{R}\,t}{\mathrm{L}}} \sin \left(n\,t - \tan^{-1} \frac{\mathrm{L}\,n}{\mathrm{R}}\right)}{\sqrt{(\mathrm{R}^2 + \mathrm{L}^2\, n^2)}},$$

by Example (139).

$$\therefore \quad \mathrm{C} = e^{-\frac{\mathrm{R}\,t}{\mathrm{L}}} \left\{ \frac{e^{\frac{\mathrm{R}\,t}{\mathrm{L}}} \sin \left\{ n\,t - \tan^{-1} \left(\frac{\mathrm{L}\,n}{\mathrm{R}}\right) \right\}}{\sqrt{(\mathrm{R}^2 + \mathrm{L}^2\, n^2)}} + k \right\}$$

by (γ); that is,

$$\mathrm{C} = \frac{\sin\,(n\,t - \theta)}{\sqrt{\mathrm{R}^2 + (\mathrm{L}\,n)^2}} + k\, e^{-\frac{\mathrm{R}\,t}{\mathrm{L}}},$$

where $\theta = \tan^{-1} \left\{ \frac{\mathrm{L}\,n}{\mathrm{R}} \right\}$.

EXAMPLES.

Integrate the following differential equations.

1. $(a + y)\, x\, d\,y + (b - x)\, y\, d\,x = 0.$

   Ans. $\log y^a\, x^b + y - x = k.$

2. $x\,y\, (a^2 + x^2)\, d\,y - (b^2 + y^2)\, d\,x = 0.$

   Ans. $(b^2 + y^2)^{a^2} (a^2 + x^2) = \mathrm{C}\, x^2.$

3. $\mathrm{Sin}\, \theta \cos \phi\, d\,\theta - \cos \theta \sin \phi\, d\,\phi = 0.$

   Ans. $\mathrm{Cos}\, \phi = k \cos \theta.$

4. $\text{Cot}\ x \operatorname{cosec}^2 y\, d y + \cot y \operatorname{cosec}^2 x\, d x = 0.$

Ans. $\text{Cot}\ x \cot y = k.$

5. $(x - y)\, d x + x\, d y = 0.$

Ans. $x = k e^{-\frac{y}{x}}.$

6. $\left(y - x \cos \frac{x}{y}\right) d y + y \cos \frac{x}{y}\, d x = 0.$

Ans. $y = k e^{-\sin \frac{x}{y}}.$

7. $(3 x - 7 y + 7)\, d y + (7 x - 3 y + 3)\, d x = 0.$

Ans. $(x - y + 1)^2 (x + y - 1)^5 = k.$

8. $t (1 - t^2) \frac{d s}{d t} + (2 t^2 - 1)\, s = n t^3.$

Ans. $s = n t + k t \sqrt{(1 - t^2)}.$

9. $\frac{d y}{d \theta} + y \cos \theta = \sin \theta \cos \theta.$

Ans. $y = \sin \theta + k e^{-\sin \theta} - 1.$

10. $\frac{d y}{d \theta} + \frac{y}{1 + \theta^2} = \frac{\tan^{-1} \theta}{1 + \theta^2}.$

Ans. $y = \tan^{-1} \theta + k e^{-\tan^{-1} \theta} - 1.$

CHAPTER XXII.

ART. 61.—Solution of Linear Equations with Constant Coefficients when the second member is 0.

There are n different values of y which satisfy the equation

$$\frac{d^n y}{d x^n} + a \frac{d^{n-1} y}{d x^{n-1}} + b \frac{d^{n-2} y}{d x^{n-2}} \dots p y = 0, \quad . \quad (a)$$

in which $a, b, \dots . p$ are constants; and if $y_1, y_2, \dots . y_n$ represent the n different values of y, then the complete value of y is

$$y = k_1 y_1 + k_2 y_2 \dots . k_n y_n,$$

that is, the complete value of y which satisfies the equation is the sum of the n distinct values of y, each multiplied by an arbitrary constant.

Rule for solving a differential equation with constant coefficients when the right-hand member is 0.

Form the *auxiliary equation* by assuming $y = k e^{\alpha x}$. Differentiate this n times, and substitute in equation (a) for example; and we have

$$k e^{\alpha x} \{ \alpha^n + a \alpha^{n-1} + b \alpha^{n-2} \dots . p \} = 0,$$

and on rejecting the common factor $k e^{\alpha x}$, we have

$$\alpha^n + a \alpha^{n-1} + b \alpha^{n-2} \dots . p = 0,$$

the *auxiliary equation.*

Let $r_1, r_2, r_3, \dots . r_n$ be the roots of this equation, all real and unequal, suppose; then the n particular solutions are

$$y = k_1 e^{r_1 x}, \quad y = k_2 e^{r_2 x} \dots ., \quad y = k_n e^{r_n x},$$

and the general solution is

$$y = k_1 e^{r_1 x} + k_2 e^{r_2 x} + k_3 e^{r_3 x} \ldots . \; k_n e^{r_n x}.$$

Again, suppose the auxiliary equation to have a pair of imaginary roots of the form $p \pm q\sqrt{-1}$; then the complete value of y will contain a pair of terms of the form

$$k_1 e^{(p + q\sqrt{-1})x} + k_2 e^{(p - q\sqrt{-1})x},$$

which may be written

$$e^{px}\{k_1 e^{(q\sqrt{-1})x} + k_2 e^{-(q\sqrt{-1})x}\} \quad . \quad . \quad (a)$$

and on expanding,

$$e^{(q\sqrt{-1})x} \quad \text{and} \quad e^{-q\sqrt{-1})x},$$

and collecting the terms we get, on substituting in (a),

$$k_1 e^{px}(\cos qx + \sqrt{-1}\sin qx) + k_2 e^{px}(\cos qx - \sqrt{-1}\sin qx)$$

$$= e^{px}\{(k_1 + k_2)\cos qx + \sqrt{-1}(k_1 - k_2)\sin qx\}$$

$$= e^{px}\{\text{M}\cos qx + \text{N}\sin qx\},$$

here $\text{M} = k_1 + k_2$ and $\text{N} = \sqrt{-1}(k_1 - k_2)$. See page 58.

Again, suppose there exist a pair of equal roots of the uxiliary equation $r_1 = r_2$ for example; then the general olution would contain a pair of terms $k_1 e^{r_1 x} + k_2 e^{r_1 x}$, which may be written $(k_1 + k_2)e^{r_1 x}$; but this would reduce the n constants to $n - 1$; therefore in order to get the general solution we assume the two roots to differ by a finite small quantity c, and find the result as c approaches 0.

Thus

$$k_1 e^{r_1 x} + k_2 e^{(r_1 + c)x} = e^{r_1 x}(k_1 + k_2 e^{cx})$$

$$= e^{r_1 x}\left(k_1 + k_2 + k_2 cx + k_2\frac{(cx)^2}{\underline{|2}} + \&\text{c.}\right),$$

and if we neglect the terms involving c^2 and higher powers it becomes

$$e^{r_1 x}(\text{M} + \text{N}x),$$

where $\text{M} = k_1 + k_2$, and $\text{N} = k_2 c$.

In like manner it may be shown that if n roots are each equal to r_1, the solution is

$$e^{r_1 x}(\mathrm{M} + \mathrm{N}\,x + \mathrm{Q}\,x^2 \;.\;.\;.\;.\; \mathrm{V}\,x^{n-1}).$$

(192.) **To solve** $\dfrac{d^2 y}{d x^2} - 10 \dfrac{d y}{d x} + 16 y = 0.$

Here the auxiliary equation is

$$a^2 - 10\,a + 16 = 0,$$

the roots of which are 8 and 2; therefore the complete solution is

$$y = k_1 e^{8x} + k_2 e^{2x},$$

the two particular solutions being

$$y = k_1 e^{8x} \quad \text{and} \quad y = k_2 e^{2x}.$$

(193.) **To solve**

$$\frac{d^2 y}{d x^2} - 6 \frac{d y}{d x} + 13 y = 0.$$

Here the auxiliary equation is

$$a^2 - 6\,a + 13 = 0,$$

the roots of which are $3 + 2\sqrt{-1}$ and $3 - 2\sqrt{-1}$; therefore the complete solution is

$$y = e^{3x}(k_1 \cos 2\,x + k_2 \sin 2\,x).$$

(194.) **To solve the equation**

$$\frac{d^5 y}{d x^5} - 4 \frac{d^4 y}{d x^4} + 10 \frac{d^3 y}{d x^3} - 12 \frac{d^2 y}{d x^2} + 5 \frac{d y}{d x} = 0.$$

Here the auxiliary equation is

$$a^5 - 4\,a^4 + 10\,a^3 - 12\,a^2 + 5\,a = 0,$$

the roots of which are 0, 1, 1, $1 - 2\sqrt{-1}$, and $1 + 2\sqrt{-1}$; therefore the complete solution is

$$y = k_1 + e^x(k_2 + k_3 x) + e^x(k_4 \cos 2\,x + k_5 \sin 2\,x).$$

(195) **To solve**

$$\frac{d^2 y}{d x^2} + a^2 y = 0.$$

Here the auxiliary equation is

$$\alpha^2 + a^2 = 0. \qquad \therefore \quad \alpha = \pm\, a \sqrt{-1};$$

therefore

$$y = k_1 \cos a x + k_2 \sin a x,$$

or

$$y = \mathrm{K} \sin (a x + \theta)$$

where $\mathrm{K} = \sqrt{k_1{}^2 + k^2{}_2}$ and $\theta = \tan^{-1} \frac{k_1}{k_2}$.

ART. 62.—The equations $\frac{d^2 y}{d x^2} + a^2 y = \sin b x$ and $\frac{d^2 y}{d x^2} + a^2 y = \cos b x$, whose right hand members are not 0, may be reduced to those whose right-hand members are 0, by differentiating and eliminating the sine or cosine function.

Given $\frac{d^2 y}{d x^2} + a^2 y = \sin b x$, on differentiating twice we have

$$\frac{d^4 y}{d x^4} + a^2 \frac{d^2 y}{d x^2} = - b^2 \sin b x,$$

and from the given equation, by substituting for $\sin b x$, we get

$$\frac{d^4 y}{d x^4} + (a^2 + b^2) \frac{d^2 y}{d x^2} + a^2 b^2 y = 0.$$

Forming the auxiliary equation, we have

$$\alpha^4 + (a^2 + b^2)\, \alpha + a^2 b^2 = 0;$$

therefore

$$\alpha = \pm\, a \sqrt{-1} \quad \text{or} \quad \alpha = \pm\, b \sqrt{-1}.$$

Hence the complete solution is

$$y = k_1\, e^{(a \sqrt{-1}) x} + k_2\, e^{(-a \sqrt{-1}) x} + k_3\, e^{(b \sqrt{-1}) x} + k_4\, e^{(-b \sqrt{-1}) x},$$

which reduces to

$$y = (k_1 + k_2) \cos a x + (k_1 - k_2) \sqrt{-1} \sin a x$$
$$+ (k_3 + k_4) \cos b x + (k_3 - k_4) \sqrt{-1} \sin b x,$$

or

$$y = \mathrm{M} \cos a x + \mathrm{N} \sin a x + \mathrm{M}_1 \cos b x + \mathrm{N}_1 \sin b x,$$

where $\mathrm{M} = k_1 + k_2$, $\mathrm{N} = (k_1 - k_2) \sqrt{-1}$, $\mathrm{M}_1 = k_3 + k_4$ and $\mathrm{N}_1 = (k_3 - k_4) \sqrt{-1}$.

EXAMPLES.

1. $\frac{d y}{d x} - a y = 0.$ *Ans.* $y = k e^{a x}.$

2. $\frac{d y}{d x} + b y = 0.$ *Ans.* $y = k e^{-b x}.$

3. $\frac{d^2 y}{d x^2} + 5 \frac{d y}{d x} + 4 y = 0.$ *Ans.* $y = k e^{-4x} + k_1 e^{-x}$

4. $\frac{d^2 y}{d x^2} + 6 \frac{d y}{d x} + 9 y = 0.$ *Ans.* $y = e^{-3x}(k + k_1 x$

5. $\frac{d^2 y}{d x^2} + 6 \frac{d y}{d x} + 13 y = 0.$

   Ans. $y = e^{-3x} \{k \cos 2 x + k_1 \sin 2 x\}.$

6. $\frac{d^3 y}{d x^3} + 3 \frac{d^2 y}{d x^2} + 3 \frac{d y}{d x} + y = 0.$

   Ans. $y = e^{-x} (k + k_1 x + k_2 x^2).$

7. $\frac{d^4 y}{d x^4} + 4 \frac{d^3 y}{d x^3} + 6 \frac{d^2 y}{d x^2} + 4 \frac{d y}{d x} + y = 0.$

   Ans. $y = e^{-x} (k + k_1 x + k_2 x^2 + k_3 x^3).$

8. $\frac{d^2 y}{d x^2} + 3 y = \sin \frac{2 \pi x}{\mathrm{T}}.$

Ans. $y = k \cos \sqrt{3}\, x + k_1 \sin \sqrt{3}\, x + k_2 \cos \frac{2 \pi x}{\mathrm{T}}$

$$+ k_3 \sin \frac{2 \pi x}{\mathrm{T}}.$$

9. $$\frac{d^2 y}{d x^2} + 4 y = \cos \frac{2 \pi x}{\mathrm{T}}.$$

Ans. $$y = k \cos 2 x + k_1 \sin 2 x + k_2 \cos \frac{2 \pi x}{\mathrm{T}}$$
$$+ k_3 \sin \frac{2 \pi x}{\mathrm{T}}.$$

10. $$\frac{d^2 x}{d t^2} + 2 f \frac{d x}{d t} + n^2 x = 0$$

when $f = 0$ . . . . . (*a*)
" $f = n$ . . . . . (*b*)
" $f < n$ . . . . . (*c*)
and " $f > n$ . . . . . (*d*)

Ans. $x = k \sin n t + k_1 \cos n t$ . . . . . (*a*)
$x = \mathrm{A}\, e^{-nt} (k + k_1 t)$ . . . . . . (*b*)
$x = \mathrm{A}\, e^{-ft} (k \cos g t + k_1 \sin g t)$ . . (*c*)

ere $g = \sqrt{n^2 - f^2}$.

$x = \mathrm{A}\, e^{-ft} (k\, e^{gt} + k_1\, e^{-gt})$ . . . . (*d*)

ere $g = \sqrt{f^2 - n^2}$.

1. $$\frac{d^5 y}{d x^5} + 2 \frac{d^2 y}{d x^2} - 9 \frac{d y}{d x} + 6 y = 0.$$

Ans. $$y = e^x (k + k_1 x) + k_2\, e^{-2x}$$
$$+ \mathrm{A} \sin \sqrt{3}\, x + \mathrm{B} \cos \sqrt{3}\, x.$$

CHAPTER XXIII.

SYMBOLICAL METHODS OF SOLVING LINEAR DIFFERENTIAL EQUATIONS.

ART. 63.—In the equation $\frac{dy}{dx} + ay = 0$, if we write θ for $\frac{d}{dx}$ it becomes $\theta y + ay = 0$, which may be written thus: $(\theta + a) y = 0$, $(\theta + a)$ denoting an operation which is to be performed on the function y.

Similarly, the equation $\frac{d^2 y}{dx^2} + (a + b)\frac{dy}{dx} + aby = 0$ may be written $\{\theta^2 + (a + b)\theta + ab\} y = 0$, or thus:

$$(\theta + a)(\theta + b) y = 0,$$

in which θ may be treated as an algebraic symbol meaning to differentiate thus:

$$(\theta + a)(\theta + b) y = (\theta + a)\left(\frac{dy}{dx} + by\right)$$

$$= \theta\left(\frac{dy}{dx} + by\right) + a\left(\frac{dy}{dx} + by\right)$$

$$= \frac{d}{dx}\frac{dy}{dx} + b\frac{dy}{dx} + a\frac{dy}{dx} + aby$$

$$= \frac{d^2 y}{dx^2} + (a + b)\frac{dy}{dx} + aby.$$

θy indicates that the function y is to be differentiated once; $\theta^2 y$ indicates that y is to be differentiated twice, &c. θy, $\theta^2 y$ are said to be *direct* operations on the function y.

If we consider the equation $(\theta + a)\,y = 0$, in which y is supposed to be known, we can determine the direct operation indicated by $(\theta + a)$ by differentiating. But the *inverse* problem arises, in which it is required to find y, so that the result of the operation on it denoted by $(\theta + a)$ may give the value cipher. If $(\theta + a)\,y = 0$, then $y = (\theta + a)^{-1}\,0$; $(\theta + a)^{-1}$ denoting the inverse operation which performed on 0 will give y.

Since the direct operation means differentiation, it follows that the inverse operation means integration; therefore $\theta^{-1}y$ means to integrate y once, $\theta^{-2}y$ means to integrate y twice, &c.

ART. 64.—**Solution of** $\dfrac{d\,y}{d\,x} + a\,y = 0$. Writing this equation in the form $y = (\theta + a)^{-1}\,0$, we know by Art. 60 that $y = \mathrm{K}\,e^{-ax}$ where K is a constant.

The student should remember that the result of the nverse operation $(\theta + a)^{-1}\,0$ is $\mathrm{K}\,e^{-ax}$, where K is an arbitrary constant. Also, if a be a function of x,

$$(\theta + a)^{-1}\,0 = \mathrm{K}\,e^{-\int a\,dx}.$$

Again, if $\dfrac{d\,y}{d\,x} + \mathrm{A}\,y = \mathrm{B}$, or thus, $y = (\theta + \mathrm{A})^{-1}\,\mathrm{B}$.

$$\therefore\quad y = e^{-\int \mathrm{A}\,dx}\left\{\int e^{\int \mathrm{A}\,dx}\,\mathrm{B}\,d\,x + k\right\},$$

where A and B are constants or functions of x. See Art. 60.

ART. 65.—**Solution of** $\dfrac{d^2\,y}{d\,x^2} + (a + b)\dfrac{d\,y}{d\,x} + a\,b\,y = \mathrm{X}$, where a and b are constants, and X a function of x. Substituting θ for $\dfrac{d}{d\,x}$ the equation becomes

$$(\theta + a)\,(\theta + b)\,y = \mathrm{X},$$

hat is,

$$y = (\theta + a)^{-1}\,(\theta + b)^{-1}\,\mathrm{X},$$

and treating $(\theta + a)^{-1}\ (\theta + b)^{-1}$ as a mere algebraic expression, by the method of partial fractions, we have

$$(\theta + a)^{-1}(\theta + b)^{-1} = \frac{1}{b-a}\left\{\frac{1}{\theta + a} - \frac{1}{\theta + b}\right\}$$

$$= \frac{1}{b-a}\left\{(\theta + a)^{-1} - (\theta + b)^{-1}\right\};$$

therefore

$$y = \frac{1}{b-a}\left\{(\theta + a)^{-1}X - (\theta + b)^{-1}X\right\}.$$

Hence

$$y = \frac{1}{b-a} e^{-ax} \qquad e^{ax} X\,dx + k\Big\}$$

$$- \frac{1}{b-a} e^{-bx}\left\{\int e^{bx} X\,dx + k^1\right\}$$

by Art. 60.

(196.) **To solve**

$$\frac{d^2y}{dx^2} - 14\frac{dy}{dx} + 13y = 0.$$

Here we have

$$(\theta - 13)(\theta - 1)\,y = 0;$$

therefore

$$y = (\theta - 13)^{-1}(\theta - 1)^{-1}0,$$

and by partial fractions

$$y = \frac{1}{12}\left\{(\theta - 13)^{-1}0 - (\theta - 1)^{-1}0\right\}.$$

$$\therefore\quad y = \frac{1}{12}\left\{k e^{13x} - k_1 e^{x}\right\}$$

by Art. 60.

(197.) **Solution of**

$$d^2y + 7\frac{dy}{dx} + 12y = \sin x.$$

Here we have

$$(\theta^2 + 7\theta + 12)\,y = \sin x,$$

or

$$y = (\theta + 3)^{-1} (\theta + 4)^{-1} \sin x,$$

and by partial fractions

$$(\theta + 3)^{-1} (\theta + 4)^{-1} = \frac{1}{\theta + 3} - \frac{1}{\theta + 4}$$
$$= (\theta + 3)^{-1} - (\theta + 4)^{-1};$$

therefore

$$y = (\theta + 3)^{-1} \sin x - (\theta + 4)^{-1} \sin x$$
$$= \frac{3 \sin x - \cos x}{10} + k e^{-3x} - \frac{4 \sin x - \cos x}{17} + k_1 e^{-4x}.$$

See examples 139 and 191.

(198.) **Solution of**

$$\frac{d^2 y}{d x^2} + a^2 y = \cos x.$$

Here

$$y = (\theta^2 + a^2)^{-1} \cos x$$
$$= \frac{1}{2 a \sqrt{-1}} \left\{ (\theta - a \sqrt{-1})^{-1} \cos x - (\theta + a \sqrt{-1})^{-1} \cos x \right\}$$
$$= \frac{1}{2 a \sqrt{-1}} \left[e^{a x \sqrt{-1}} \left\{ \int e^{-a x \sqrt{-1}} \cos x \, d x + k \right\} \right]$$
$$- \frac{1}{2 a \sqrt{-1}} \left[e^{-a x \sqrt{-1}} \left\{ \int e^{a x \sqrt{-1}} \cos x \, d x + k_1 \right\} \right], \text{ \&c.}$$

(199.) Suppose we require the result of the operation

$$\frac{a + b \theta + c \theta^2 + d \theta^3 + e \theta^4}{a' + b' \theta + c' \theta^2 + d' \theta^3 + e' \theta^4} \text{ upon } \sin k t.$$

Now θ means, differentiate $\sin k t$ once; θ^2 means, differentiate $\sin k t$ twice, &c.; therefore

$$\theta \sin k t = k \cos k t,$$
$$\theta^2 \sin k t = - k^2 \sin k t,$$
$$\theta^3 \sin k t = - \theta k^2 \sin k t = - k^3 \cos k t,$$
$$\theta^4 \sin k t = k^4 \sin k t,$$

therefore it will be seen that the expression reduces to

$$\frac{(a - c\,k^2 + e\,k^4) + (b - d\,k^2)\,\theta}{(a' - c'\,k^2 + e'\,k^4) + (b' - k^2\,d')\,\theta}\sin k\,t$$

$$= \frac{\alpha + \beta\,\theta}{\alpha' + \beta'\,\theta}\sin k\,t$$

where

$$\alpha = a - c\,k^2 + e\,k^4, \qquad \beta = b - d\,k^2.$$
$$\alpha' = a' - c'\,k^2 + e'\,k^4, \qquad \beta' = b' - d'\,k^2.$$

Again,

$$(\alpha + \beta\,\theta)\sin k\,t = \alpha\sin k\,t + \beta\,k\cos k\,t$$
$$= \sqrt{\alpha^2 + (\beta\,k)^2}\,\sin(k\,t + \theta),$$

where $\tan\theta = \dfrac{\beta\,k}{\alpha}$.

This is the effect of the direct operation. The inverse operation

$$(\alpha' + \beta'\,\theta)^{-1}\sin k\,t = \frac{1}{\sqrt{\alpha'^2 + (\beta'\,k)^2}}\sin(kt - \phi) + c\,e^{-\frac{\alpha' t}{\beta'}},$$

where $\tan\phi = \dfrac{\beta'\,k}{\alpha'}$ and c is a constant; therefore the complete result of the above operation on $\sin k\,t$ is

$$\sqrt{\frac{\alpha^2 + (\beta\,k)^2}{(\alpha')^2 + (\beta'\,k)^2}}\sin\{k\,t + \theta - \phi\} + c\,e^{-\frac{\alpha' t}{\beta'}}.$$

$c\,e^{-\frac{\alpha' t}{\beta'}}$ is said to be the evanescent term where t stands for time, since its value diminishes as time increases.

For the working out of the inverse operation see examples 197 and 139.

Let us notice the effect of the operation denoted by θ upon $\sin 2\,\pi\,n\,t$, where $n = 100$, $\pi = 3{\cdot}14$, say.

Now

$$\theta\sin 2\,\pi\,n\,t = \theta\sin 628\,t = 628\cos 628\,t$$

$$= 628\sin\left(628\,t + \frac{\pi}{2}\right),$$

that is, the effect of θ upon the sine function $\sin 628\,t$ is to multiply it by 628, and alter its phase by $\frac{\pi}{2}$, or one quarter period.

EXAMPLES.

Solve by the symbolical method the following examples.

1. $\frac{d^2 y}{d x^2} - 3\frac{d y}{d x} + 2y = 0.$ *Ans.* $y = k e^x + k' e^{2x}$.

2. $\frac{d^2 y}{d x^2} - 4\frac{d y}{d x} + 13y = 0.$

   Ans. $y = k e^{2x} \cos 3x + k' e^{2x} \sin 3x$.

3. $\frac{d^2 y}{d x^2} - 2\frac{d y}{d x} + y = 0.$ *Ans.* $y = e^x (k + k' x)$.

4. $\frac{d^3 y}{d x^3} - 3\frac{d^2 y}{d x^2} + 3\frac{d y}{d x} - y = 0.$

   Ans. $y = e^x (k + k' x + k'' x^2)$.

5. $\frac{d^3 y}{d x^3} - \frac{d^2 y}{d x^2} - \frac{d y}{d x} + y = 0.$

   Ans. $y = k e^{-x} + e^x (k' + k'' x)$.

6. $\frac{d^4 y}{d x^4} + 2a^2 \frac{d^2 y}{d x^2} + a^4 y = 0.$

   Ans. $y = (k + k' x) \cos ax + (k'' + k''' x) \sin ax$.

7. $\frac{d^2 y}{d x^2} + a^2 y = \cos ax.$

   Ans. $y = k \cos ax + \left(k' + \frac{x}{2a}\right) \sin ax$.

8. $\frac{d^2 y}{d x^2} - 2a\frac{d y}{d x} + a^2 y = e^x.$

   Ans. $y = (k + k' x) e^{ax} + \frac{e^x}{(a - 1)^2}$.

CHAPTER XXIV.

APPLICATION OF THE CALCULUS TO SOME MECHANICAL PROBLEMS.

(200.) **To find the law connecting the tension and angle of lapping of a belt on a pulley.**

Let O be the centre of the pulley, A O B the angle of lapping of the belt on the pulley, α, T_1 the tension of the belt at A, and T the tension at B, $T > T_1$.

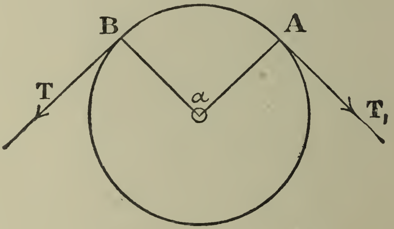


Fig. 37.

Consider a very short portion of the belt between A and B. Let the tension at one end of it be t, then the tension at the other end of it will be $t + d\,t$. Let R denote the reaction of the pulley. These three forces keep the small portion of the belt in equilibrium, therefore they are proportional to the three sides of a triangle. Draw the triangle of forces. We have the angle between the two forces t and $t + d\,t$ equal to

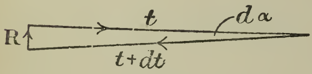


Fig. 38.

the angle between the two radii at the extremities of the small portion under consideration. Denote this angle by $d\alpha$ and we have

$$R = t\,d\alpha.$$

But the friction is $= \mu R$, where μ is the coefficient of friction between the belt and the pulley; that is, $dt = \mu R = \mu t\,d\alpha$. Therefore

$$\frac{dt}{t} = \mu\,d\alpha.$$

Integrating, we have

$$\log t = \mu\alpha + k.$$

Now $\alpha = 0$ where $t = T_1$.

$$\therefore \log T_1 = k.$$

$$\therefore \log t = \mu\alpha + \log T_1.$$

That is,

$$\log \frac{t}{T_1} = \mu\alpha,$$

or

$$\frac{t}{T_1} = e^{\mu\alpha}.$$

Therefore

$$t = T_1 e^{\mu\alpha}.$$

This gives the tension at any point when T_1, μ, and α are given, when the belt is about to slip.

(201.) **To find the law for the thickness of a long pump-rod of uniform strength.**

Let a denote its cross-section in square inches at any point, f its safe stress; therefore its safe load

$$w = af \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

At a height dh above this point the load is $w + acdh$ where c is the weight of a cubic inch of the material of the rod; therefore, for uniform strength we must have

$$w + acdh = f(a + da) \quad . \quad . \quad . \quad (2)$$

Subtracting (1) from (2) we have

$$a\,c\,d\,h = f\,d\,a,$$

that is,

$$\frac{d\,a}{a} = \frac{c}{f}\,d\,h.$$

Integrating, we get

$$\log a = \frac{c}{f}\,h + \text{C}.$$

Where $h = 0$, that is, at the lower end of the rod, let $a = a_0$.

$$\therefore \quad \text{C} = \log a_0.$$

$$\therefore \quad \log \frac{a}{a_0} = \frac{c\,h}{f}.$$

$$\therefore \quad a = a_0\,e^{\frac{c\,h}{f}}.$$

This law determines the cross-section at any height h above its lower end.

Suppose $a_0 = {\cdot}8$, $c = {\cdot}28$, $h = 300$, and $f = 1000$.

$$\therefore \quad a = {\cdot}8\,(2{\cdot}718)^{{\cdot}084}.$$

$$\log a = \bar{1}{\cdot}9031 + {\cdot}084 \times {\cdot}4343$$

$$= \bar{1}{\cdot}9369.$$

$$\therefore \quad a = {\cdot}8702 \text{ sq. inch.}$$

(202.) **To find the shape and deflection of a semi-girder of uniform section loaded at its extremity.**

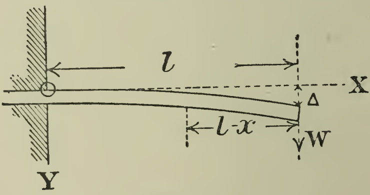


Fig. 39.

Let l denote the length of the beam in inches, W the load in lbs. at its extremity, 0 the origin, 0 X the axis of X, and 0 Y the axis of Y.

At a distance x from the end, the bending moment M, produced by W, is

$$M = (l - x)\,W;$$

but the bending moment at a section is

$$M = \frac{E\,I}{R},$$

where E is the modulus of elasticity of the material of the beam, I the moment of inertia of the section about a horizontal line through its centre of area, and R is the radius of curvature of the girder at the section.

By Art. 32

$$R = \frac{\left\{1 + \left(\frac{d\,y}{d\,x}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2\,y}{d\,x^2}},$$

and since, in the case of beams, $\frac{d\,y}{d\,x}$ is so small that we may neglect it, therefore we have

$$\frac{1}{R} = \frac{d^2\,y}{d\,x^2};$$

therefore

$$E\,I\,\frac{d^2\,y}{d\,x^2} = (l - x)\,W.$$

On integrating this differential equation once, we have

$$E\,I\frac{d\,y}{d\,x} = \left(l\,x - \frac{x^2}{2}\right)W + \text{constant}.$$

Now $\frac{d\,y}{d\,x} = 0$ where $x = 0$, therefore the constant $= 0$; therefore we have

$$E\,I\frac{d\,y}{d\,x} = \left(l\,x - \frac{x^2}{2}\right)W,$$

which gives the slope of the girder for a given value of x.

Integrate again, and we have

$$\mathrm{E\,I}\,y = \left(\frac{l\,x^2}{2} - \frac{x^3}{6}\right)\mathrm{W}.$$

No constant is added, since $y = 0$ when $x = 0$.

$$\therefore \quad y = \left(\frac{l\,x^2}{2} - \frac{x^3}{6}\right)\frac{\mathrm{W}}{\mathrm{E\,I}}.$$

This gives the deflection y for any given value of x. The value of y is greatest where $x = l$, therefore greatest deflection

$$\Delta = \frac{\mathrm{W}\,l^3}{3\,\mathrm{E\,I}}.$$

(203.) **To find the shape and deflection of a semi-girder of uniform section loaded uniformly with a load of w lb. per foot run.**

Let l denote the length of the semi-girder in feet, the axis of X and Y, as in the preceding example.

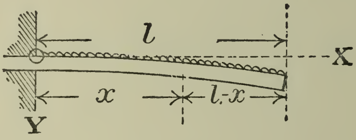


FIG. 40.

The bending moment at a distance $l - x$ from its extremity is

$$\mathrm{M} = w\,(l - x)\,\frac{(l - x)}{2},$$

for the weight $w\,(l - x)$ acts as if it were collected at a distance $\frac{l - x}{2}$ from the section, therefore

$$\mathrm{E\,I}\,\frac{d^2\,y}{d\,x^2} = \frac{w}{2}\,(l - x)^2,$$

Integrating, we have

$$\mathrm{E\,I}\frac{d\,y}{d\,x} = \frac{w}{2}\left(l^2 x - l\,x^2 + \frac{x^3}{3}\right).$$

No constant is added, since $\frac{d\,y}{d\,x} = 0$, where $x = 0$. This equation gives the slope, $\frac{d\,y}{d\,x}$, of the girder for a given value of x. Integrate again, and we have

$$\mathrm{E\,I}\,y = \frac{w}{2}\left(\frac{l^2 x^2}{2} - \frac{l\,x^3}{3} + \frac{x^4}{12}\right).$$

Here, again, no constant is added, since $y = 0$ where $x = 0$. This equation gives the deflection for a given value of x. The greatest deflection, Δ, is where $x = l$.

$$\therefore\quad \Delta = \frac{w\,l^4}{8\,\mathrm{E\,I}} = \frac{\mathrm{W}\,l^3}{8\,\mathrm{E\,I}}$$

where $\mathrm{W} = w\,l$ the total load.

To get Δ in inches we must take l in inches, and the unit of length must be one inch in calculating I.

(204.) **To find the shape and deflection of a girder of uniform section loaded at its centre and supported at its ends.**

Let l denote the distance between the two supports, W the load at the centre. Suppose one half of this girder to

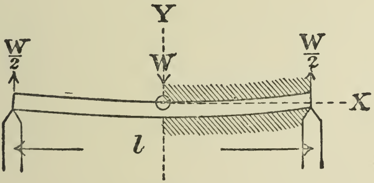


FIG. 41.

become embedded in a wall, then the shape of the beam will be unaltered, and the problem reduces to 202; but instead of W we have $\frac{W}{2}$, and instead of l we have $\frac{l}{2}$; therefore the equation of the deflection curve is

$$\mathrm{E\,I}\,y = \left\{\left(\frac{l}{2}\right)\frac{x^2}{2} - \frac{x^3}{6}\right\}\frac{W}{2},$$

where y is the height above and x the distance from the centre of the beam.

The greatest deflection is equal to the greatest value of y, that is, where $x = \frac{l}{2}$.

$$\therefore\ \Delta = \frac{W\,l^3}{48\,\mathrm{E\,I}}.$$

(205.) **To find the shape and the deflection of a girder of uniform section supported at its ends and loaded uniformly.**

Let l denote the length of the girder in feet, w the load per foot run. Taking the middle point 0 as origin, the horizontal line through 0 as the axis of X, and the vertical line through 0 as the axis of Y, we have the bending moment at a distance x from the origin,

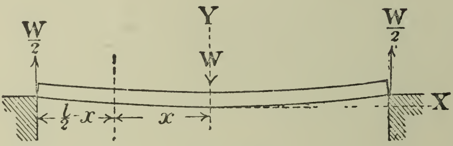


FIG. 42.

$$\mathrm{M} = \frac{w\,l}{2}\left(\frac{l}{2} - x\right) - \frac{w}{2}\left(\frac{l}{2} - x\right)^2 = \frac{w}{2}\left(\frac{l^2}{4} - x^2\right);$$

therefore

$$\mathrm{E\,I}\,\frac{d^2 y}{d\,x^2} = \frac{w}{2}\left(\frac{l^2}{4} - x^2\right).$$

Integrating, we have

$$\mathrm{E\,I}\frac{dy}{dx} = \frac{w}{2}\left(\frac{l^2 x}{4} - \frac{x^3}{3}\right).$$

No constant is added, since $\frac{dy}{dx} = 0$ where $x = 0$.

Integrate again; therefore

$$\mathrm{E\,I}\,y = \frac{w}{2}\left(\frac{l^2 x^2}{8} - \frac{x^4}{12}\right).$$

There is no constant added, since $y = 0$ when $x = 0$.

The deflection is equal to the greatest value of y; that is, where $x = \frac{l}{2}$; therefore

$$\Delta = \frac{5\,w\,l^4}{384\,\mathrm{E\,I}} = \frac{5\,\mathrm{W}\,l^3}{384\,\mathrm{E\,I}}, \text{ where } \mathrm{W} = w\,l.$$

(206.) **To find the shape and deflection of a girder of uniform section fixed at its ends and loaded at its centre.**

Let l denote the length of the girder in feet, W the load at its centre in lbs.

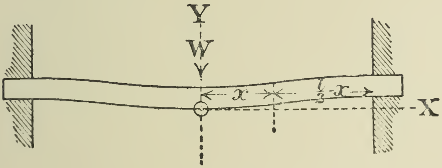


FIG. 43.

Taking the middle point 0 as origin and axis, as in the preceding example, we have the bending moment at a distance x from the centre.

$$(a)\quad \mathrm{M} = \frac{\mathrm{W}}{2}\left(\frac{l}{2} - x\right) - \mathrm{M}',$$

where M′ is the bending moment due to the stresses in the girder at its intersection with the wall. Therefore

$$\mathrm{E\,I}\frac{d^2 y}{d x^2} = \frac{\mathrm{W}}{2}\left(\frac{l}{2} - x\right) - \mathrm{M}'.$$

Integrating, we have

$$(b) \quad \mathrm{E\,I}\frac{d y}{d x} = \frac{\mathrm{W}}{2}\left(\frac{l x}{2} - \frac{x^2}{2}\right) - \mathrm{M}'x.$$

There is no constant to be added, since $\frac{d y}{d x} = 0$ where $x = 0$.

We have also $\frac{d y}{d x} = 0$ where $x = \frac{l}{2}$, and this gives $\mathrm{M}' = \frac{\mathrm{W}\,l}{8}$.

The bending moment at the centre is got from (a) by substituting 0 for x, and $\frac{\mathrm{W}\,l}{8}$ for M′; therefore

$$\mathrm{M} = \frac{\mathrm{W}\,l}{4} - \frac{\mathrm{W}\,l}{8} = \frac{\mathrm{W}\,l}{8}.$$

Integrating (b), we have

$$(c) \quad \mathrm{E\,I}\,y = \frac{\mathrm{W}}{2}\left(\frac{l x^2}{4} - \frac{x^3}{6}\right) - \frac{\mathrm{W}\,l x^2}{16}.$$

The constant is 0, since $y = 0$ where $x = 0$.

The greatest deflection is got by substituting $\frac{l}{2}$ for x in equation (c); therefore

$$\Delta = \frac{\mathrm{W}\,l^3}{192\,\mathrm{E\,I}}.$$

The points of inflection may be found from (a) by substituting 0 for M; tha

$$0 = \frac{\mathrm{W}}{2}\left(\frac{l}{2} - x\right) - \frac{\mathrm{W}\,l}{8};$$

Therefore

$$x = \frac{l}{4},$$

that is, the middle segment is one-half the length of the girder.

(207.) **To find the shape and the deflection of a girder of uniform section fixed at its extremities and loaded uniformly.**

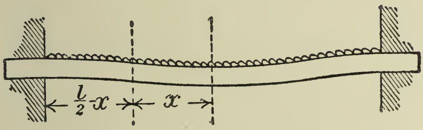


FIG. 44.

Let l denote the length of the girder in feet, w the load per foot run, the axes as in the preceding example. The bending moment at a distance x from the middle is

$$M = \frac{w\,l}{2}\left(\frac{l}{2} - x\right) - \frac{w}{2}\left(\frac{l}{2} - x\right)^2 - M'. \quad . \quad (a)$$

where M′ is the bending moment due to the stresses in the girder at its intersection with the wall; therefore

$$E\,I\frac{d^2\,y}{d\,x^2} = \frac{w}{2}\left(\frac{l^2}{4} - x^2\right) - M' \quad . \quad . \quad . \quad (b)$$

Integrating, we have

$$E\,I\frac{d\,y}{d\,x} = \frac{w}{2}\left(\frac{l^2\,x}{4} - \frac{x^3}{3}\right) - M'\,x \,. \quad . \quad . \quad (c)$$

The constant being 0, since $\frac{d\,y}{d\,x} = 0$ where $x = 0$.

Again, where

$$x = \frac{l}{2},\ \frac{d\,y}{d\,x} = 0,$$

therefore

$$M' = \frac{W\,l}{12} \text{ where } W = w\,l.$$

Substituting in (c) for M' we have, on integrating,

$$E\,I\,y = \frac{w}{2}\left(\frac{l\,x^2}{8} - \frac{x^4}{12}\right) - \frac{W\,l\,x^2}{24},$$

the constant being 0, since $y = 0$ where $x = 0$.

The greatest deflection is got from this equation by substituting $\frac{l}{2}$ for x.

Hence

$$\Delta = \frac{W\,l^3}{384\,E\,I}.$$

The bending moment M_c at the centre is got from (a) by substituting 0 for x and $\frac{W\,l}{12}$ for M'; that is,

$$M_c = \frac{w\,l}{2} \times \frac{l}{2} - \frac{w}{2} \times \left(\frac{l}{2}\right)^2 - \frac{W\,l}{12}.$$

$$\therefore \quad M_c = \frac{W\,l}{24}.$$

That is, the bending moment at the centre is half that a the intersection of the beam with the wall, or the beam s twice as ready to break at its intersection with the wall as t its centre.

The points of inflection may be found from (a) by maki $M = 0$. Therefore

$$\frac{w}{2}\left(\frac{l^2}{4} - x^2\right) = \frac{w\,l^2}{12}.$$

Hence $x = \pm \cdot 288\,l$; therefore the middle segm. $= \cdot 576\,l$.

(208.) **To investigate the rule for the strength c a thick cylindric pipe.**

Let Fig. 45 denote a section of a thick cylindric pipe whose internal and external radii are r and R respectively, and let the internal pressure be p' lbs. per square inch and the external pressure q lbs. per square inch, which may be neglected, the safe tensile stress of the material f'.

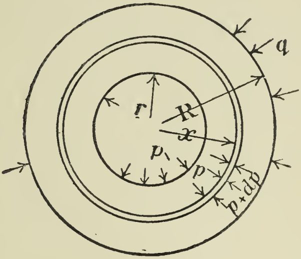


FIG. 45.

Consider a thin cylindric coaxial shell of the pipe of radius x and thickness $d\,x$, and let the internal pressure on this shell be p lb. per square inch, and the external pressure $p + d\,p$ lb. per square inch.

The former pressure tends to burst the shell and the latter tends to crush it. Denoting the tensile stress in this shell by f, we have

$$2\,p\,x - 2\,(p + dp)\,(x + d\,x) = 2\,f\,d\,x,$$

and neglecting the differential of the second order, we have

$$(f + p)\,d\,x + x\,d\,p = 0. \quad . \quad . \quad . \quad (a)$$

Again, assuming that a plane section, before the pressure acts, remains a plane section under pressure, we have

$$p - f = c, \text{ a constant};$$

therefore $f = p - c$. Substituting for f in equation (a), we get

$$(2\,p - c)\,d\,x + x\,d\,p = 0,$$

that is,

$$\frac{d\,x}{x} + \frac{d\,p}{2\,p - c} = 0.$$

Integrating, we have

$$\log x + \tfrac{1}{2}\log(2\,p - c) = \text{constant};$$

therefore

$$2\log x + \log(2\,p - c) = \text{constant}.$$

Hence $x^2(2p - c) = 2A$ where A is a constant;

$$\therefore \quad p = \frac{A}{x^2} + \frac{c}{2}.$$

This law shows how the pressure diminishes as x increases. Where $x = r$, $p = p'$, and neglecting the external pressure, where $x = R$, $p = 0$; therefore

$$0 = \frac{A}{R^2} + \frac{c}{2}, \text{ and } p' = \frac{A}{r^2} + \frac{c}{2}.$$

Hence

$$A = \frac{p' R^2 r^2}{R^2 - r^2}, \text{ and } \frac{c}{2} = -\frac{p' r^2}{R^2 - r^2}.$$

But $f' = p' - c$; therefore

$$f' = p' + \frac{2p' r^2}{R^2 - r^2}.$$

Therefore

$$\frac{f'}{p'} = \frac{R^2 + r^2}{R^2 - r^2};$$

or thus

$$\frac{R}{r} = \sqrt{\frac{f' + p'}{f' - p'}}.$$

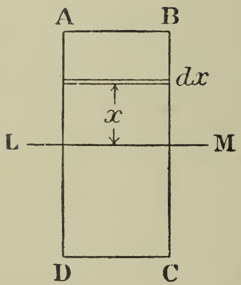


FIG. 46.

(209.) **To find the moment of inertia of a rectangle about a line, parallel to one of its sides, passing through its centre of area.**

The moment of inertia of an area about a line is the sum of each element of the area multiplied by the square of its distance from the line.

Let A B C D (Fig. 46) be a rectangle, L M a line through its centre parallel to A B.

Let $AB = b$ and $BC = d$.

Suppose the area split up into an indefinite number of ips, parallel to A B, whose breadth is dx; then an element of area is $b\,d\,x$. The moment of inertia of this element about L M is $x^2\,b\,d\,x$ where x is its distance from L M; therefore the moment of inertia I, of the rectangle about L M, is

$$I = \int_{-\frac{d}{2}}^{+\frac{d}{2}} b\,x^2\,d\,x = b\left[\frac{x^3}{3}\right]_{-\frac{d}{2}}^{+\frac{d}{2}}.$$

$$\therefore\quad I = \frac{b\,d^3}{12}.$$

(210.) **To find the moment of inertia of a circle about its centre, and also about a diameter.**

Let Fig. 47 represent a circle whose radius is r and whose centre is at the origin.

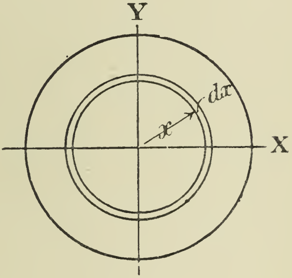


FIG. 47.

Suppose the area split up into an indefinite number of concentric strips, the breadth of each being $d\,x$, and let the radius of one of these strips be x, then its area is $2\,\pi\,x\,d\,x$, and its moment of inertia about the centre is $2\,\pi\,x^3\,d\,x$; therefore the moment of inertia of the whole circle about its centre is

$$I = 2\,\pi\int_0^r x^3\,d\,x = 2\,\pi\left[\frac{x^4}{4}\right]_0^r.$$

$$\therefore\quad I = \frac{\pi\,r^4}{2} = \pi\,r^2 \times \frac{r^2}{2} = A\,k^2$$

where A is its area, and k is the *radius of gyration* which is $\frac{r}{\sqrt{2}}$.

The moment of inertia of the circle about a diameter is half its moment of inertia about its centre, since

$$r^2 = x^2 + y^2$$

where r is the distance of any element of area a from the origin,

Now

$$a\,r^2 = a\,x^2 + a\,y^2,$$

$$\therefore\quad \Sigma\,a\,r^2 = \Sigma\,a\,x^2 + \Sigma\,a\,y^2,$$

that is, I_c about centre is equal to I_x about axis of X plus I_y about axis of Y; but it is evident that

$$\Sigma\,a\,x^2 = \Sigma\,a\,y^2.$$

$$\therefore\quad I_x = \tfrac{1}{2}\,I_c.$$

$$\therefore\quad I_x = \frac{\pi\,r^4}{4} = \frac{\pi\,d^4}{64}$$

where d is the diameter.

(211.) **To find the moment of inertia of a cylinder about its axis.**

The moment of inertia of a mass about any line is the sum of all such terms as each little element of mass multiplied by the square of its distance from the line. Let l denote the length of the cylinder, r the radius of its end, ρ its density, therefore its mass is $\dfrac{\pi\,r^2\,l\,\rho}{g}$, and since by 210 the radius of gyration is $\dfrac{r}{\sqrt{2}}$, it follows that

$$I = \frac{\pi\,r^2\,l\,\rho}{g}\,\frac{r^2}{2} = \text{mass} \times \frac{r^2}{2}.$$

(212.) **To find the moment of inertia of a sphere of mass, M, about its centre, and also about a diameter.**

Let r denote the radius of the sphere, and suppose i divided up into an indefinite number of concentric shells o thickness $d\,x$. Let the radius of any one of these shells be ; its mass is therefore

$$\frac{4\,\pi\,\rho\,x^2\,d\,x}{g},$$

and its moment of inertia about the centre is

$$\frac{4\,\pi\,\rho\,x^4\,d\,x}{g},$$

therefore the moment of inertia of the sphere about its centre is

$$\mathrm{I}_c = \frac{4\,\pi\,\rho}{g}\int_0^r x^4\,d\,x = \frac{4\,\pi\,r^3\,\rho}{g} \times \frac{r^2}{5},$$

that is,

$$\mathrm{I}_c = \frac{4}{3}\,\frac{\pi\,r^3\,\rho}{g} \times \frac{3\,r^2}{5} = \frac{3}{5}\,\mathrm{M}\,r^2.$$

The moment of inertia, I_d, of a sphere about a diameter is two-thirds of its moment of inertia about its centre.

For

$$\Sigma\,m\,x^2 = \Sigma\,m\,y^2 = \Sigma\,m\,z^2,$$

and

$$\Sigma\,m\,x^2 + \Sigma\,m\,y^2 + \Sigma\,m\,z^2 = \Sigma\,m\,r^2,$$

where m is an element of mass; therefore

$$\Sigma\,m\,x^2 + \Sigma\,m\,y^2 = \frac{2}{3}\,\Sigma\,m\,r^2,$$

that is, the moment of inertia about the axis of Z is $\frac{2}{3}$ of I_c.

$$\therefore \quad \mathrm{I}_d = \frac{2}{3} \times \frac{3}{5}\,\mathrm{M}\,r^2 = \frac{2}{5}\,\mathrm{M}\,r^2.$$

CHAPTER XXV.

APPLICATION OF THE CALCULUS TO SOME ELECTRICAL PROBLEMS.

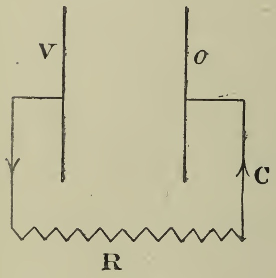


FIG. 48.

(213.) **To find the law for the discharge of a condenser through a resistance with no self-induction.**

Let K denote the capacity of the condenser, V its potential, R the resistance in the circuit, and Q the quantity of electricity; then

$$Q = K V, \quad \text{and} \quad C = \frac{V}{R},$$

and since the current diminishes as time increases, we have

$$\frac{dQ}{dt} = -C = \quad K\frac{dV}{dt} = -\frac{V}{R};$$

therefore

$$\frac{dV}{V} = -\frac{dt}{KR},$$

and on integrating we have

$$\log V = -\frac{t}{KR} + k,$$

where k is a constant.

Let $V = V_0$ when $t = 0$; therefore

$$k = \log V_0.$$

$$\therefore \quad \log \frac{V}{V_0} = -\frac{t}{KR},$$

or

$$V = V_0 e^{-\frac{t}{KR}}. \quad . \quad . \quad . \quad . \qquad (\alpha)$$

If we want to find the resistance, R, we have

$$R = \frac{t}{K \log_e \frac{V_0}{V}},$$

or

$$R = \frac{t}{2 \cdot 302 \, K \log_{10} \frac{V_0}{V}}. \quad . \quad . \quad . \qquad (\beta)$$

(α) gives the voltage at any time, t, when V_0, R and K are given. (β) gives the resistance when t, K V and V_0 are given.

(214.) **To find the law for the growth of a current in a circuit when the voltage suddenly changes from zero to V, taking into account the self-induction in the circuit.**

Let R denote the resistance in the circuit, L the self-induction, C the current, and t the time in seconds. Then

$$V = RC + L \frac{dC}{dt}.$$

This equation may be written thus

$$\frac{dC}{\frac{V}{R} - C} = \frac{R\,dt}{L},$$

and on integrating we have

$$\log\left(\frac{V}{R} - C\right) = -\frac{Rt}{L} + k,$$

where k is a constant.

Let $C = 0$ when $t = 0$; therefore

$$k = \log \frac{V}{R}.$$

Hence

$$\log\left(\frac{V}{R} - C\right) - \log \frac{V}{R} = -\frac{R\,t}{L},$$

or

$$\log\left(1 - \frac{R\,C}{V}\right) = -\frac{R\,t}{L};$$

therefore

$$1 - \frac{R\,C}{V} = e^{-\frac{R\,t}{L}};$$

therefore

$$C = \frac{V}{R}\left(1 - e^{-\frac{R\,t}{L}}\right),$$

Helmholtz's equation.

This law will give the current at any time, t seconds after the voltage suddenly changes from zero to V.

If $t = \infty$ the law becomes

$$C = \frac{V}{R} \text{ Ohm's law,}$$

since $e^{-\frac{R\,t}{L}} = 0$ when $t = \infty$.

(215.) **To destroy the effect of self-induction in a circuit by the introduction of a condenser shunt.**

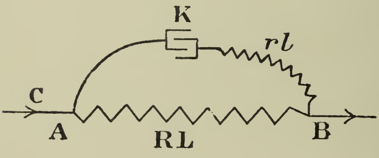


Fig. 49.

Suppose we have a portion of a line A B to signal through, and we want the whole arrangement, from A to B. to act as a mere resistance. Let V denote the potential difference between A and B, R the resistance in the main circuit A B, L the self-induction. Let l be the self-induction and r the resistance in the condenser shunt.

The total current

$$C = \frac{V}{\text{combined resistance}}.$$

The reciprocal of the combined resistance

$$= \frac{1}{R + L\,\theta} + \frac{1}{r + l\,\theta + \dfrac{1}{K\,\theta}} \text{ where } \theta = \frac{d}{d\,t}$$

and K is the capacity of the condenser.

$$\therefore\quad C = \left(\frac{1}{R + L\,\theta} + \frac{1}{r + l\theta + \dfrac{1}{K\,\theta}}\right)V,$$

that is,

$$C = \frac{1 + (r\,K + R\,K)\,\theta + (l\,K + L\,K)\,\theta^2}{R + (L + R\,r\,K)\,\theta + (R\,l\,K + L\,r\,K)\,\theta^2 + L\,l\,K\,\theta^3}\,V.$$

This operation performed on the voltage gives the current. But we want Ohm's Law, viz. $C = \frac{V}{R}$. We want no self-induction or capacity effects, therefore the coefficient of V must be equal to $\frac{1}{R}$.

Equate $\frac{1}{R}$ and the coefficient, therefore

$$\frac{1}{R} = \frac{1 + K\,(r + R)\,\theta + K\,(l + L)\,\theta^2}{R + (L + R\,r\,K)\,\theta + K\,(R\,l + r\,L)\,\theta^2 + L\,l\,K\,\theta^3}.$$

Clearing of fractions and equating the coefficients of like powers of θ, we have

$$R\,K\,(r + R) = L + R\,r\,K, \quad . \quad . \quad . \quad (a)$$

$$R\,K\,(l + L) = K\,(R\,l + L\,r), \quad . \quad . \quad (b)$$

and

$$L\,l\,K = 0. \quad . \quad . \quad . \quad . \quad . \quad (c)$$

Consider (c) L and K are not 0; therefore $l = 0$, that is, there must be no self-induction in the condenser branch.

Again, if $l = 0$, from equation (b) we get $R = r$, and from equation (a) we get

$$K = \frac{L}{R^2}.$$

That is, the resistance in the main branch must be equal to the resistance in the condenser branch, and the capacity of the condenser must be equal to the self-induction in the main branch divided by the square of its resistance.

Suppose the voltage obeys the law

$$V = V_0 \sin p\,t,$$

we have

$$C = \frac{1 + K(R + r)\theta + K(l + L)\theta^2}{R + (L + RKr)\theta + K(Rl + rL)\theta^2 + LlK\theta^3} V_0 \sin p\,t$$

$$= \frac{\{1 - p^2(L + l)K\} + K(R + r)\theta}{\{R - p^2(Rl + Lr)K\} + \{(L + RKr) - p^2 LlK\}\theta} V_0 \sin p\,t$$

$$= \frac{m + n\,\theta}{m' + n'\,\theta} V_0 \sin p\,t, \text{ say.}$$

$$\therefore \quad C = \sqrt{\frac{m^2 + p^2 n^2}{m'^2 + p^2 n'^2}}\, V_0 \sin\left\{p\,t + \tan^{-1}\left(\frac{p\,n}{m}\right) - \tan^{-1}\left(\frac{p\,n'}{m'}\right)\right\}.$$

See Example 199.

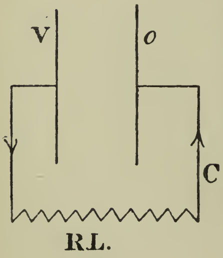


FIG. 50.

(216.) **To find the law for the discharge of a condenser of capacity K, through a resistance R, with self-induction L, in the circuit.**

Let V be the potential difference, R the resistance, L the self-induction, and C the current.

We have

$$V = RC + L\frac{dC}{dt} \quad . \quad (a)$$

Also,

$$Q = KV,$$

Q being the quantity of electricity.

Now

$$C = -\frac{dQ}{dt} = -K\frac{dV}{dt};$$

therefore

$$\frac{dC}{dt} = -K\frac{d^2V}{dt^2}.$$

Substituting in (a) we get

$$V = -RK\frac{dV}{dt} - LK\frac{d^2V}{dt^2};$$

therefore

$$\frac{d^2V}{dt^2} + \frac{R}{L}\frac{dV}{dt} + \frac{1}{LK}V = 0 \quad . \quad . \quad . \quad (b)$$

Assume

$$V = k e^{xt}.$$

$$\therefore \quad \frac{dV}{dt} = k x e^{xt},$$

and

$$\frac{d^2V}{dt^2} = k x^2 e^{xt};$$

therefore substituting in (b) we get

$$x^2 + \frac{R}{L}x + \frac{1}{LK} = 0,$$

the auxiliary equation.

$$\therefore \quad x = -\frac{R}{2L} \pm \sqrt{\left(\frac{R^2}{4L^2} - \frac{1}{LK}\right)},$$

that is,

$$x = -\frac{R}{2L} \pm \sqrt{\left(\frac{1}{LK} - \frac{R^2}{4L^2}\right)}\sqrt{-1} = -\alpha \pm \beta\sqrt{-1},$$

where

$$\alpha = \frac{R}{2L}, \quad \text{and } \beta = \sqrt{\left(\frac{1}{LK} - \frac{R^2}{4L^2}\right)}.$$

If $\frac{1}{LK} - \frac{R^2}{4L^2}$ be positive, the solution of equation (*b*) is therefore

$$V = e^{-at}(A_{,} \cos \beta t + A_2 \sin \beta t). \quad \text{Art. 61.}$$

When $t = 0$, let $V = V_0$; therefore $A_{,} = V_0$.

When $t = 0$, let $C = 0$;

therefore

$$A_2 = \frac{a V_0}{\beta}.$$

$$\therefore \quad V = V_0 e^{-at}\left(\cos \beta t + \frac{a}{\beta} \sin \beta t\right),$$

that is

$$V = \frac{V_0}{\beta} \sqrt{(a^2 + \beta^2)}\, e^{-at} \sin(\beta t + \theta) \quad . \quad (1)$$

where $\tan \theta = \frac{\beta}{a}$.

If $\frac{R}{2L} = 0$ the solution is

$$V = A_{,} \cos \beta t + A_2 \sin \beta t.$$

Let $V = V_0$ when $t = 0$, and let $C = 0$ when $t = 0$;

$$\therefore \quad A_2 = 0, \text{ and } A_1 = V_0;$$

therefore

$$V = V_0 \cos \beta t \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If $\frac{1}{LK} = \frac{R^2}{4L^2}$ the two roots are equal, and the solution is

$$V = (A + Bt)\, e^{-at}.$$

When

$$t = 0 \text{ let } V = V_0 \quad \therefore \quad A = V_0,$$

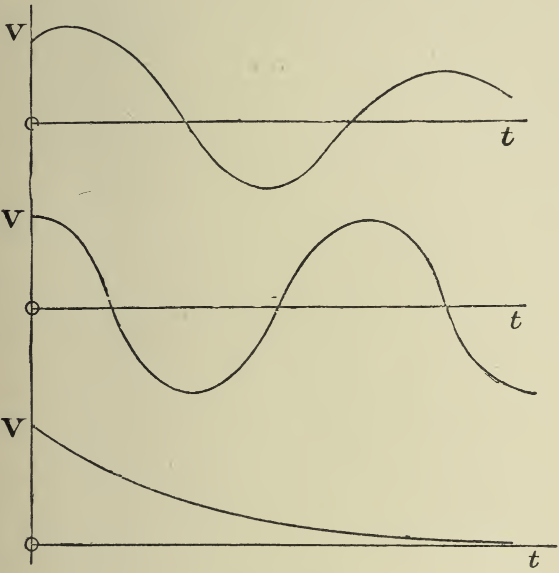
and when

$$t = 0 \text{ let } C = 0 \quad \therefore \quad B = a V_0;$$

therefore

$$V = V_0(1 + at)\, e^{-at} \quad . \quad . \quad . \quad . \quad (3)$$

If V and t in equation (1) be plotted on squared paper, t horizontally and V vertically, we shall have a curve as shown in Fig. 51, which represents a damped vibration with diminishing amplitude.



FIGS. 51 to 53.

If V and t in equation (2) be plotted as in equation (1), we shall have a curve as shown in Fig. 52, which represents an undamped vibration with undiminishing amplitude.

If V and t in equation (3) be similarly plotted, we shall have a curve as shown in Fig. 53, which shows that the condenser is leaking.

(217.) **The sum of the averages of the volts across two resistances is greater than the average volts between the two ends of those resistances.**

Let V denote difference of potential between A and B (Fig. 54), R the resistance, and L the coefficient of self-induction; v the potential between B and F, r and l the resistance and self-induction respectively between B and F.

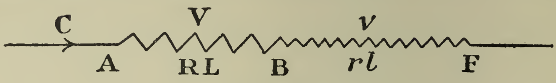


FIG. 54.

At any instant the total voltage between A and F is equal to

$$\{R + r + (L + l)\,\theta\}\,C,$$

where C is the current.

And

$$V = (R + L\,\theta)\,C,$$

and

$$v = (r + l\,\theta)\,C.$$

Let $C = a \sin p\,t$; therefore the average current is equal to $\dfrac{a}{\sqrt{2}}$.

Therefore average

$$V = \frac{a}{\sqrt{2}}\sqrt{R^2 + L^2\,p^2},$$

and the average

$$v = \frac{a}{\sqrt{2}}\sqrt{r^2 + l^2\,p^2}.$$

Also the average

$$(V + v) = \frac{a}{\sqrt{2}}\sqrt{\{(R + r)^2 + (L + l)^2\,p^2\}}.$$

It can be easily shown by algebra that

$$\sqrt{\{(R + r)^2 + (L + l)^2\,p^2\}} < \sqrt{(R^2 + L^2\,p^2)} + \sqrt{(r^2 + l^2\,p^2)}$$

therefore the sum of the averages of the volts across two resistances is greater than the average volts between the two ends of those resistances.

(218.) **To diminish the idle current in the primary coil of a transformer by means of a condenser shunt.**

Let K denote the capacity of the condenser, c the current in the condenser branch, V the voltage, C, L, and R the

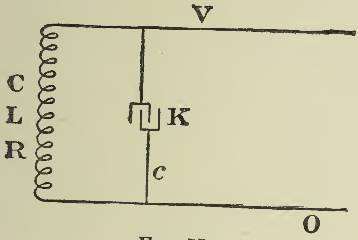


FIG. 55.

rrent, self-induction and resistance respectively, in the imary coil of the transformer.

Let $V = a \sin p t$.

Now

$$c = K \theta V$$

nd

$$C = \frac{V}{R + L \theta},$$

$$\therefore \quad C + c = \left(K \theta + \frac{1}{R + L \theta}\right) a \sin p t;$$

at is,

$$C + c = \left(\frac{1 + R K \theta + L K \theta^2}{R + L \theta}\right) a \sin p t.$$

Operating on $a \sin p t$ we get

$$C + c = a \sqrt{\frac{(1 - L K p^2)^2 + R^2 K^2 p^2}{R^2 + L^2 p^2}} \sin (p t + \theta - \phi),$$

ere $\theta = \tan^{-1}\left(\frac{R K p}{1 - L K p^2}\right)$, and $\phi = \tan^{-1}\left(\frac{L p}{R}\right)$.

The amplitude $a \sqrt{\frac{(1 - L K p^2)^2 + R^2 K^2 p^2}{R^2 + L^2 p^2}}$ is less than if K is sufficiently small, R being small, and is a minimum hen $K = \frac{L}{(L p)^2 + R^2} = s$, say.

For given values of L, p and R, if $(C + c)$ and K be plotted, we shall get a curve as shown in the accompanying figure, which shows how $C + c$ diminishes as K increases,

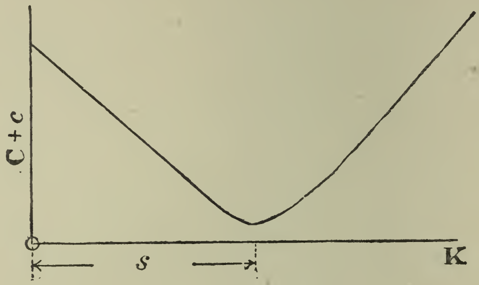


FIG. 56.

and when $K = s$, $C + c$ is a minimum. As K increases beyond s, $C + c$ increases almost directly proportional to $K - s$.

(219.) **The Ferranti Effect.** If E be the voltage of the generator, C the current, ρ the resistance of the lamps, K

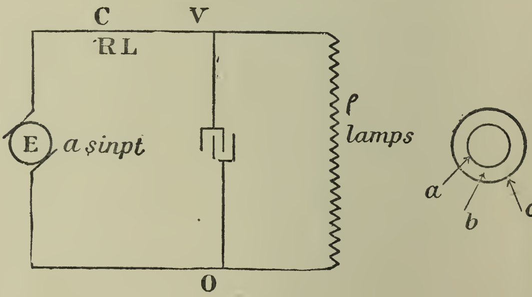


FIG. 57.

the capacity of the leads, acting as a condenser, L the self-induction, and R the resistance, a being the going lead, c the returning lead, and b a dielectric; then the voltage V is sometimes greater than E.

Now

$$C = \frac{V}{\frac{1}{K\theta}} + \frac{V}{\rho} = V\left(K\theta + \frac{1}{\rho}\right),$$

and

$$E - C(R + L\theta) = V,$$

where $C(R + L\theta)$ is the drop of volts in the leads.

$$\therefore\quad E - V\left(K\theta + \frac{1}{\rho}\right)(R + L\theta) = V.$$

$$\therefore\quad V = \frac{E}{1 + (R + L\theta)\left(K\theta + \frac{1}{\rho}\right)}.$$

This denominator must be less than unity in order to make V greater than E.

The denominator $= 1 + \frac{R}{\rho} + \theta\left(\frac{L}{\rho} + RK\right) + LK\theta^2$, and the effect of this inverse operation on a sine function $a \sin pt$ is

$$V = \frac{a}{\sqrt{\left(1 + \frac{R}{\rho} - p^2 LK\right)^2 + p^2\left(\frac{L}{\rho} + RK\right)^2}} \sin(pt - \theta'),$$

where

$$\theta' = \tan^{-1}\left\{\frac{p\left(\frac{L}{\rho} + RK\right)}{1 + \frac{R}{\rho} - p^2 LK}\right\}.$$

If $K = 0$,

$$V = \frac{a}{\sqrt{\left(1 + \frac{R}{\rho}\right)^2 + \left(\frac{pL}{\rho}\right)^2}} \sin(pt - \theta_1)$$

here

$$\theta_1 = \tan^{-1}\left(\frac{pL}{\rho + R}\right).$$

The effective voltage, if there is no condenser effect, is

$$= \frac{a}{\sqrt{\left(1 + \frac{R}{\rho}\right) + \left(\frac{p\,L}{\rho}\right)^2}},$$

and the effective voltage with a condenser effect is

$$= \frac{a}{\sqrt{\left(1 + \frac{R}{\rho} - p^2 L K\right)^2 + p^2\left(\frac{L}{\rho} + R K\right)^2}}.$$

K may have such a value as will render the latter effective voltage greater than the former, the condition being that $\frac{2\,L}{K} > R^2 + p^2 L^2$ when $\rho = \propto$.

MISCELLANEOUS EXAMPLES.

1. What is a differential coefficient? Give illustrations from velocity, from slope of curves, &c.

2. Prove the rule for differentiating a quotient, also fo x^m, sin x, and log x.

3. Prove that $\frac{d\,u}{d\,x} = \frac{d\,u}{d\,y}\,\frac{d\,y}{d\,x}$.

4. Differentiate log (log x), $e^{a\,x}$ sin $r\,x$, $\frac{(1 - x^2)^{\frac{3}{2}} \sin^{-1}}{x}$ x^{x^n}, and e^{x^x}.

5. If $x = e^{\frac{x-y}{y}}$, find $\frac{d\,y}{d\,x}$.

6. Prove Taylor's Theorem. Give two examples of its us

7. State Maclaurin's Theorem. Expand a^x and sin x.

8. If $y = \sin k\,x$, find $\frac{d\,y}{d\,x}$ from first principles; also fi the differential coefficient of a product.

9. Differentiate $\frac{x}{\sqrt{a^2 + x^2} - x}$, $\sqrt{1 - x^2}$, $\sin^{-1} x$, a $x^{\sin x}$.

10. Find $\frac{d^5 y}{d x^5}$ if $y = \frac{1 + x}{1 - x}$.

11. Expand $\sin (x + a)$ in powers of a by Taylor's Theorem, and afterwards put $x = 0$.

12. Find the value of $\frac{e^x - e^{-x}}{\log (1 + x)}$ when $x = 0$.

13. If c be the hypotenuse of a right-angled triangle, find the lengths of the other sides when the area of the triangle is a maximum.

14. If $y = x^n$, find $\frac{d^m y}{d x^m}$ $m < n$.

15. Describe the meaning of $d u = \left(\frac{d u}{d x}\right) d x + \left(\frac{d u}{d y}\right) d y$.

16. If $u = \log \tan \frac{y}{x}$, prove that $\frac{d^2 u}{d x \, d y} = \frac{d^2 u}{d y \, d x}$.

17. Find the value of $\frac{x - \sin^{-1} x}{\sin^3 x}$ when $x = 0$.

18. If u is a function of y and z, and these are both functions of x, prove that

$$\frac{d u}{d x} = \left(\frac{d u}{d y}\right) \frac{d y}{d x} + \left(\frac{d u}{d z}\right) \frac{d z}{d x}.$$

Explain the meaning of this equation. Give illustrations. If $u = z^2 + y^2 + z y$, and $z = \sin x$, $y = e^x$, find $\frac{d u}{d x}$ in two ways.

19. Prove the rules for finding maxima and minima values of a function. Illustrate by a curve.

20. Find the strongest rectangular beam which can be cut from a cylindric tree. Find also the stiffest beam.

21. If the motion of a piston worked by a crank of length r, and connecting rod of length l, is such that its distance x, from the end of its stroke when the crank makes an angle θ with the line of centre, is approximately $x = r \, (1 - \cos \theta)$

$+ \frac{r^2}{4\,l}(1 - \cos 2\,\theta)$, find the acceleration when $\theta = 30°$, $r = 1$ ft., $l = 5$ ft.

22. Find the limiting value of $\frac{x - 1}{x^n - 1}$ when $x = 1$.

23. Prove the expressions for the *subtangent*, *subnormal*, and *intercepts* on the axes by the tangent at any point to a curve.

24. Find the equations of the tangent and normal to a curve at any point. Find these for the point (1, 6) on the curve $y^2 = 36x$.

25. Expand a^x by Maclaurin's Theorem. In your answe put $a = e$.

26. Expand $\log(1 + x)$ and $\tan^{-1} x$.

27. Find the true value of $\frac{\tan x - x}{x - \sin x}$ when $x = 0$.

28. Divide 100 into two parts, so that five times the square of one part and three times the square of the other part shall be a minimum.

29. If $V = R\,C + \cdot 01\,\frac{d\,C}{d\,t}$, and if $C = 50 \sin \frac{2\,\pi\,t}{\cdot 03}$, find V, and roughly indicate the nature of C and V by curves.

30. Find the area of an indicator diagram if the law of expansion is $p\,v =$ const. Find p_e in terms of p_1, p_3, and r, and the amount of work per cubic foot of steam, p_e being effective pressure, p_1 initial pressure, and p_3 back pressure. Find r for a maximum amount of work per cubic foot of steam, assuming no condensation.

31. Find the expressions for the position of the centre and the radius of curvature of any curve at any point.

32. Find the radius of curvature at the vertex of the parabola $y = m\,x^2$.

33. Write out the integrals of x^m, $\frac{1}{x - a}$, $\frac{1}{a^2 + x^2}$, e^x, $\frac{1}{\sqrt{x^2 + a^2}}$, $\frac{1}{(x - a)(x - b)}$, $f'(x) \div f(x)$.

34. What is the rule for integrating by parts? Apply it in the integration of $\sqrt{a^2 - x^2}$.

34*a*. Find the equation to the cycloid. Find the length of the curve.

35. Compare the volumes of a paraboloid of revolution, a cone and a cylinder on the same base and of the same height.

36. Find the shape taken by a flexible chain loaded uniformly horizontally.

37. If $V = R C + L \frac{d C}{d t}$, and $C = a \sin \frac{2 \pi t}{T}$, find V.

38. Find the area of the hypothetical indicator diagram, the law of the expansion being $p v^k =$ const., and afterwards the work done per cubic foot of steam. Cut off at $\frac{1}{r}$ of the stroke.

39. Find the moment of inertia of a cylinder about its axis.

40. Integrate

$$(x^2 + 7 x + 12)^{-1}, \quad (1 + x + x^2)^{-1}, \quad e^{a x} \sin m x,$$
$$\sqrt{a^2 - x^2}, \text{ and } \sin m x \cos n x.$$

41. Explain how we determine the most economical electrical conductor (*a*) for a constant current, (*b*) to deliver a definite amount of power at a distant place.

42. 120,000 watts are to be delivered 3 miles away; dynamo voltage 1000. Find the proper conductor. Compare the answer with answer given by Lord Kelvin's rule.

43. A straight uniform beam is fixed at one end and loaded at the other; find its shape when loaded.

44. Find $\int_0^{2\pi} \sin^2 k x \, d x$, $\int_0^{2\pi} \sin k x \cos r x \, d x$.

45. Investigate the equation to the catenary. Find the length of an arc of it, and its area.

46. Define $\sin h x$, $\cos h x$, $\tan h x$, $\sin h^{-1} x$ and $\cos h^{-1} x$.

47. Integrate

$$(1 - 2 x + 2 x^2)^{-1}, \quad (1 + x - x^2)^{-1}, \quad (a + b \cos \theta)^{-1},$$
$$x \log x, \quad \text{and } x^2 e^x .$$

48. Find the area of the segment of an ellipse.

49. Find the surface generated by the revolution of

$$y = \frac{c}{2}\left\{e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right\}$$

about the axis of X.

50. If $V = a + b \sin n t$, find C when $V = R C + L \frac{d C}{d t}$. Give the complete value of C. Why do we not usually give the complete value?

51. Neglecting evanescent terms, given $V = a \sin n t$, find the current in a circuit which has resistance, self-induction and capacity.

52. Prove the rules for the development of a function in Fourier's Series.

53. If $C = a_0 + a_1 \sin p t + b_1 \cos p t + a_2 \sin 2 p t + b_2 \cos 2 p t +$ &c., what does a meter measure?

54. Solve $\frac{d^2 x}{d t^2} + a \frac{d x}{d t} + b x = 0$. Show how, as a increases from 0, the nature of the solution changes. Apply to any dynamical or electrical problem.

55. State exactly what is meant by $M\, d x + N\, d y$ being a complete differential. What is the test? Why?

56. Integrate $x y (1 + x^2) d y - (1 + y^2) d x = 0$,

$$x\, d y - y\, d x - \sqrt{x^2 + y^2}\, d x = 0.$$

57. Integrate

$$x (1 - x^2) \frac{d y}{d x} + (2 x^2 - 1) y = b x^3.$$

$$\frac{d^2 y}{d x^2} - 7 \frac{d y}{d x} + 12 y = 0.$$

58. Solve $\theta^2 v + f \theta v + n^2 v = 0$, showing the various kinds of solution for different values of f and n. Apply the solution either to the discharging of a condenser or to the damping of vibrations of a body.

59. Prove the rule for destroying the effects due to self-induction in a branch of a network of conductors.

60. Explain what is meant by $C^2 r + \frac{t^2}{r}$ in electrical problems. If C is constant, find the condition of maximum economy. What is t usually, and on what assumption?

61. Operate with

$$\frac{a + b\,\theta + c\,\theta^2 + d\,\theta^3 + e\,\theta^4}{a' + b'\,\theta + c'\,\theta^2 + d'\,\theta^3 + e'\,\theta^4}$$

upon $A \sin kt$; θ means $\frac{d}{dt}$.

62. Find r and K so as to destroy effects of self-induction, for all outside circuits. Current following any law whatever.

63. Develop the periodic function

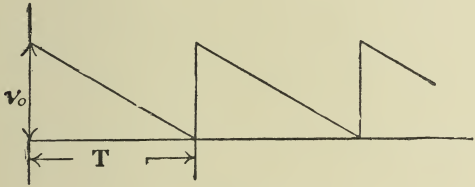


FIG. 58.

64. Integrate

$$\int \frac{dx}{1 + x - x^2}, \quad \int \frac{dx}{x^2 + 4x + 5}, \quad \int \frac{dx}{\sqrt{x^2 - ax}},$$

$$\int \frac{dx}{ax - x^2}, \quad \int e^{ax} \sqrt{\sin mx}\, dx, \quad \text{and} \quad \int e^x \sqrt{x}\, dx.$$

65. Find the area of a loop of the curve $r^2 = a^2 \cos 3\theta$.

66. Find the moment of inertia of a rectangle about a line through its centre parallel to one side.

67. Find the centre of area of a sector of a circle.

68. Find the law of tension in a belt on a pulley when slipping is occurring.

69. Find the law of leakage from an electric condenser through a constant resistance.

70. When is $\phi(x, y)\,d\,x + \psi(x, y)\,d\,y$ a complete differential?

71. When is $x \cos x$ a maximum or minimum?

72. Integrate (1) $\sin x \cos y\,d\,x - \cos x \sin y\,d\,y = 0$.

$$(2)\ \frac{d\,y}{d\,x} + \cos x\,y = \sin x \cos x.$$

$$(3)\ (y - x)\,d\,y + y\,d\,x = 0.$$

$$(4)\ \frac{d^4\,y}{d\,x^4} + 2\frac{d^2\,y}{d\,x^2} + y = 0.$$

73. Investigate the motion of a body vibrating under the action of a frictional resistance.

74. What is the meaning of

$$\left(\frac{d}{d\,x} - a\right)^{-1}\left(\frac{d}{d\,x} - b\right)^{-1} \mathrm{X}.$$

Illustrate by an example.

75. Find the tangent, normal, subtangent and subnormal to $x^m\,y^n = a$.

76. In the curve whose equation is $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, show that the part of the tangent intercepted between the axes is a.

77. $\displaystyle\int\frac{d\,x}{x^2 - a^2}$, $\displaystyle\int x^2\sqrt{a + x}\,d\,x$, $\displaystyle\int\frac{x^3\,d\,x}{(x-1)^2\,(x^2+1)}$.

78. Find the volume and surface of a paraboloid of revolution bounded by a section at right angles to the axis.

79. Find the moment of inertia of an ellipse about its major axis.

80. Prove that the moment of inertia of a body about any axis is equal to its moment of inertia about a parallel axis through its centre of mass, plus the whole mass multiplied by the square of the distance between the axes.

81. Find the moment of inertia of a circle (a) about its centre, (b) about a diameter.

82. Find the moment of inertia of a sphere (a) about its centre, (b) about a diameter.

83. Find the moment of inertia of a right cylinder about its axis.

84. Solve $(2x + 4y + 7)\,dx + (x + 7y + 1)\,dy = 0$.

85. Neglecting evanescent terms, given $V = a \sin nt$, find the current in a circuit which has resistance R, capacity K, and self-induction L.

86. Develop the periodic function.

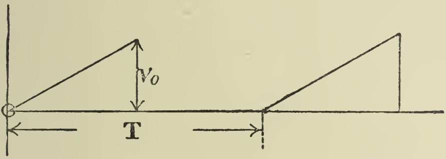


FIG. 59.

87. Express the motion of the piston of a steam engine approximately as two simple harmonic motions.

88. With the hypothetic law, work per cubic foot of steam $= 144\, r \left\{ p \left(\frac{1 + \log r}{r} \right) - p_3 \right\} - x$. If Mr. Willans' best law of cut-off, $r = \frac{p'}{p_3 + 10}$, is correct, find how x depends on r.

89. Show that when a curve is very *flat*, $\frac{d^2 y}{d x^2}$ is its *curvature*.

90. Solve $\frac{d^2 y}{d x^2} + f \frac{d y}{d x} + 25 y = 0$,

1st. When f is greater than 10, say $f = 12$.
2nd. When f is equal to 10.
3rd. When f is less than 10, say $f = 8$, or 6, or 4, or 2, or 0.

91. Plot the three kinds of curves you obtain in Example 90.

92. Find the area of a cycloid, the radius of the generating circle being r.

93. Find expressions for the position of the centre o curvature of a curve.

94. For what curves is it true that the subnormal i constant?

95. Expand $\log(x + h)$, and put $x = 1$.

96. Integrate $(x^2 - 36)^{-1}$, $\sin^2 x$, and $\cos^5 x$.

97. If $y = x(20 - x)$, for what value of x will y have it greatest value? If $\frac{dy}{dx} = ky$, show that $y = a e^{kx}$.

98. Show by actually plotting points on squared pape that if we add the ordinates of $y = a \sin x$, and $y = b \cos x$, we obtain the curve $y = \sqrt{a^2 + b^2} \sin(x + e)$, where $e = \tan^{-1}\frac{b}{a}$.

99. Write in words the meaning of

$$\frac{d^2 s}{d t^2} = - a s; \quad \frac{d^2 s}{d t^2} + f \frac{d s}{d t} + a s = 0.$$

100. Find the maximum and minimum values of

$$\frac{(a + x)(b + x)}{x}, \quad \frac{9x^2 + 6x - 7}{12x - 8}, \text{ and } \frac{a^2}{x} + \frac{b^2}{a - x}.$$

101. Find two factors of a, so that the sum of their squares is a minimum.

102. Solve $(3x - 7y + 7)\,dy + (7x - 3y + 3)\,dx = 0$.

103. If x is the distance of the piston of a steam engine from mid-stroke, θ, angle of crank from dead point, r length of crank, neglecting angularity of connecting rod, $x = r \cos \theta$. Take $\theta = \frac{2 \pi t}{T}$, where T = time in seconds of one revolutio If weight of piston is 200 lb., $r = 1$ foot. Make a diagra showing forces acting on piston rod due to acceleration piston.

104. If l is length of connecting rod, the motion of piston may be taken as very nearly being one simple harmonic motion plus a much smaller one of half the period. Or $x = r \cos \theta - \frac{r^2}{4l}(1 - \cos 2\theta)$. Now find the diagram o

accelerating forces if $l = 4$ feet. Observe that the over-tone or "kick" becomes more important in the acceleration than it was in the mere motion.

105. Find the subnormal and subtangent of the curve whose equation is $y^n = a^{n-1}x$.

106. Show that if $s = a\,e^{-pt}\sin nt$, then

$$\frac{d^2 s}{dt^2} + 2p\frac{ds}{dt} + (p^2 + n^2)s = 0.$$

107. When the rate of increase of a quantity is proportional to the quantity, or

$$\frac{dy}{dx} = ky,$$

show that

$$y = a\,e^{kx}.$$

This has been called the compound interest law. Show that it is the law for the sections of long pump-rods, &c.; law for leaking electric condensers; law for belts slipping on pulleys; law for atmospheric pressure as one changes in level; and many other cases.

108. If

$$\frac{d^2 y}{dx^2} + n^2 y = 0,$$

$$y = \mathrm{A}\sin nx + \mathrm{B}\cos nx,$$

where A and B are any constants whatever.

109. Find the area of the curve $y = a\,x^n$ between the ordinates $x = x'$ and $x = x_2$. Show the shape of this curve when $n = 2$, when $n = \frac{1}{2}$, and when $n = -1{\cdot}1$. The answer is indeterminate when $n = -1$. What is then the area?

110. Find the volume and surface of a sphere.

111. Integrate $\tan x$, $\dfrac{x^3}{a + bx^4}$, $\dfrac{x}{\sqrt{1 - x^2}}$, $\dfrac{1}{a + bx^2}$, $\dfrac{(1 + x^2)^2}{x^2}$, $\dfrac{x - 2}{x\sqrt{x}}$, $\tan^2 x$, $\sqrt{\dfrac{a + x}{a - x}}$, $x\sqrt{x + a}$, $\dfrac{1}{\sqrt{x + a} + \sqrt{x}}$, $\dfrac{x^3}{\sqrt{a^8 - x^8}}$, $\dfrac{1}{x \log x}$, and $\dfrac{x^2}{(a + bx)^3}$.

112. Integrate $\frac{1}{x^2 + p\,x + q}$; show that we obtain quite different answers for certain values of p and q.

113. Integrate $\frac{a\,x + b}{x^2 + p\,x + q}$, $(5\,x^2 + 4\,x + 8)^{-1}$, and compare $\int \frac{d\,x}{1 - a^2\,x^2}$ and $\int \frac{d\,x}{1 + a^2\,x^2}$.

114. Integrate by parts—

$$x \log x, \quad \sqrt{a^2 - x^2}, \quad \text{and} \quad x^5\, e^{a\,x}.$$

115. If $s = a + b\,t + c\,t^2$, find velocity and acceleration. What are a, b, and c? If $s = a \sin p\,t$, find velocity and acceleration. What do a and p mean? Deduce the rule for finding the periodic time in simply vibrating bodies.

116. Find the result of the operation—

$$\frac{\mathrm{A} + \mathrm{B}\,\theta + \mathrm{C}\,\theta^2 + \mathrm{D}\,\theta^3}{a + b\,\theta + c\,\theta + d\,\theta^3}$$

upon $m \sin p\,t$. Use this in finding the current in a circuit whose E.M.F. is $m \sin p\,t$, with resistance R, self-induction L, and having in series a condenser of capacity K. If there is no E.M.F., find the current (that is, find the surging current when the condenser is discharged through this circuit).

117. Prove Taylor's Theorem. Expand $\sin x$ by Maclaurin's Theorem.

118. Illustrate $\frac{d^2\,u}{d\,x\,d\,y} = \frac{d^2\,u}{d\,y\,d\,x}$ by taking $u = \log\left(\tan \frac{y}{x}\right)$.

119. What are the rules for finding maxima and minima values of a function of one variable? Illustrate by curves. The volume of a cylinder being fixed, find its height h and radius r when the sum of the areas of its convex surface and one end is a minimum.

120. Find the tangent through the point (1, 5) to the curve $y = a\,x^{\frac{1}{2}}$.

121. Find the curvature at the origin in the curve $y = a\,x^{\frac{1}{2}}$.

122. A straight beam of uniform section, fixed at one end and loaded uniformly. Find its shape.

123. Integrate

$$2\frac{d^2 y}{d x^2} - \frac{d y}{d x} - 15 y = 0,$$

and

$$\frac{d^2 y}{d x^2} + 2\frac{d y}{d x} + 5 y = 0$$

124. A man sitting in a train says: "We are now going at sixty miles an hour." What does he mean? Illustrate your answer by a curve.

125. Find the square root of mean square of $a \sin p t$.

126. Compare the areas of a parabolic segment, a rectangle, and a triangle on the same base and with same height.

127. Prove that a chain loaded uniformly horizontally is a parabolic curve.

128. What curves have their subtangents $= x^2$?

129. Find by integration the centre of gravity of a segment of a paraboloid of revolution whose base is perpendicular to its axis.

130. Find the moment of inertia of a circular cylinder about a diameter of one of its ends.

131. Integrate $\dfrac{3x - 1}{x^3 + x^2 - 2x}$ and $\dfrac{1}{a + bx + cx^2}$.

132. Integrate

$$\frac{1 + x}{1 + y} + \frac{d x}{d y} = 0, \qquad y + \sqrt{x^2 + y^2} = x\frac{d y}{d x}.$$

133. State the rule for solving linear differential equations with constant coefficients. Find y if

$$(\theta - \alpha)(\theta - \beta)(\theta - \gamma) y = 0.$$

134. Find the volume and surface generated by the revolution of an equilateral triangle about a line passing through its vertex parallel to its base.

135. If the law of expansion of steam in the cylinder of a

steam engine be $p\,v^k =$ const., the work done per stroke in foot-pounds is

$$a\,l\left\{p'\,\frac{k\,r^{-1} - r^{-k}}{k-1} - p_3\right\},$$

where l is the stroke in feet, a the cross-section of the piston in square inches, p' the initial pressure, p_3 the back pressure, the cut-off taking place at one rth of the stroke. Prove this. What is the true value when $k = 1$?

136. Find the volume generated by a cycloid revolving about its base.

137. Find the moment of inertia of an ellipse about a diameter.

138. Find the shape assumed by a uniformly heavy telegraph wire hanging between two points of support.

139. Solve $\dfrac{d^2 y}{d x^2} + l^2 y = \cos m x$.

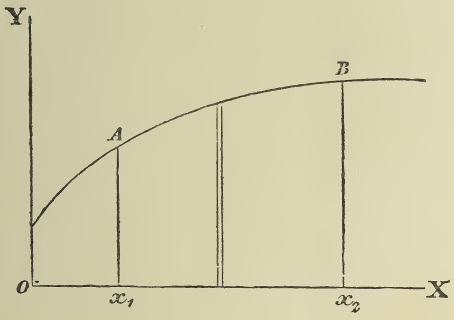
140. Solve $\dfrac{d^4 y}{d x^4} - 2a\dfrac{d y}{d x} + a^2 y = e^x$.

APPENDIX.

Centre of a Plane Area and Centre of Mass.

HE sum of the moments of all the elements of any plane ea about a line in its plane is equal to the moment of the ole area about the same line.

In the accompanying figure it is required to find the ntre of the area A x_1 x_2 B. Let the co-ordinates of the ntre of area be $\bar{x}$ and $\bar{y}$.



Suppose the area to be split up into an indefinite number strips parallel to the axis of Y, the breadth of each being x. Let the height of any one of these strips, at a distance from the axis of Y, be y; then the area of this strip will $y\,d\,x$. The whole area

$$A\,x_1\,x_2\,B = \int_{x_1}^{x_2} y\,d\,x.$$

The moment of an element about the axis of y is $x\,y\,d\,x$, and the sum of the moments of all the elements is

$$\int_{x_1}^{x_2} x\,y\,d\,x,$$

and the moment of the whole area is

$$\bar{x}\int_{x_1}^{x_2} y\,d\,x,$$

therefore

$$\bar{x} = \frac{\int_{x_1}^{\bar{x}_2} x\,y\,d\,x}{\int_{x_1}^{x_2} y\,d\,x} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This is a general formula for the $\bar{x}$ of the centre of any plane area in rectangular co-ordinates.

To get $\bar{y}$ we take moments about the axis of X, therefore

$$\bar{y}\int_{x_1}^{x_2} y\,d\,x = \int_{x_1}^{x_2} \tfrac{1}{2}\,y^2\,d\,x\,;$$

therefore

$$\bar{y} = \frac{\frac{1}{2}\int_{x_1}^{x_2} y^2\,d\,x}{\int_{x_1}^{x_2} y\,d\,x} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

a general formula for the $\bar{y}$ of the centre of any plane area.

EXAMPLE.

Find the centre of area of the parabola $y^2 = 4\,x$ between the limits $x = 0$ and $x = 10$.

Here

$$\bar{x} = \frac{\int_0^{10} x\,y\,d\,x}{\int_0^{10} y\,d\,x},$$

and

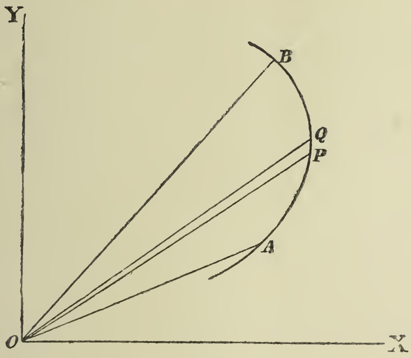
$$y = 2\,x^{\frac{1}{2}}.$$

$$\therefore \quad \bar{x} = \frac{2\displaystyle\int_0^{10} x^{\frac{3}{2}}\,d\,x}{2\displaystyle\int_0^{10} x^{\frac{1}{2}}\,d\,x} = \frac{\frac{4}{5}\,x^{\frac{5}{2}}}{\frac{4}{3}\,x^{\frac{3}{2}}} = \frac{3}{5}\Big[\,x\,\Big]_0^{10}$$

$$\therefore \quad \bar{x} = \frac{3}{5} \times 10 = 6.$$

$$\bar{y} = \frac{\frac{1}{2}\displaystyle\int_0^{10} 4\,x\,d\,x}{2\displaystyle\int_0^{10} x^{\frac{1}{2}}\,d\,x} = \frac{\Big[\,x^2\,\Big]_0^{10}}{\frac{4}{3}\Big[\,x^{\frac{3}{2}}\,\Big]_0^{10}} = \frac{3}{4}\,x^{\frac{1}{2}} = \frac{3}{4}\sqrt{10}.$$

If the equation of the curve be given in polar co-ordinates where the origin is the pole and the axis of X is the initial ine, and we require the $\bar{x}$ and $\bar{y}$ of the area O A B.



Imagine the area O A B divided up into an indefinite number of triangles whose common vertex is O, one of them being O P Q. The area of O P Q is $\frac{1}{2} r^2\,d\,\theta$ where $r = \mathrm{O\,P}$ and θ is the vectorial angle of P.

The x of the centre of area of O P Q is $\frac{2}{3} r \cos\theta$, and tl of it is $\frac{2}{3} r \sin\theta$

Hence

$$\bar{x} = \frac{\int \frac{2}{3} r \cos\theta \, \frac{1}{2} r^2 \, d\theta}{\int \frac{1}{2} r^2 \, d\theta},$$

that is,

$$\bar{x} = \frac{\frac{2}{3}\int r^3 \cos\theta \, d\theta}{\int r^2 \, d\theta} \quad . \quad . \quad . \quad . \quad (3$$

between proper limits.

Similarly,

$$\bar{y} = \frac{\frac{2}{3}\int r^3 \sin\theta \, d\theta}{\int r^2 \, d\theta} \quad . \quad . \quad . \quad . \quad (4$$

(3) and (4) are general formulæ for the $\bar{x}$ and $\bar{y}$ of centre of any plane curve whose polar equation is given.

EXAMPLE.

Find the centre of area enclosed by the curve wl equation is

$$r = a(1 + \cos\theta).$$

This curve is symmetrical about the axis of X, co quently $\bar{y} = 0$.

$$\bar{x} = \frac{\frac{2}{3}\int_0^{\pi} a^3 (1 + \cos\theta)^3 \cos\theta \, d\theta}{\int_0^{\pi} a^2 (1 + \cos\theta)^2 \, d\theta} = \frac{2}{3} a \frac{\int_0^{\pi} (1 + \cos\theta)^3 \cos\theta}{\int_0^{\pi} (1 + \cos\theta)^2 \, d}$$

herefore

$$\frac{2}{3}a\frac{\left[\sin\theta+\frac{3\theta}{2}+\frac{3\sin 2\theta}{4}+3\sin\theta-\sin^3\theta+\frac{3\theta}{8}+\frac{3}{16}\sin 2\theta+\frac{\sin\theta\cos^3\theta}{4}\right]_0^\pi}{\left[\theta+2\sin\theta+\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right]_0^\pi},$$

$$\therefore\quad \bar{x}=\frac{2}{3}a\times\frac{5}{4}=\frac{5a}{6}.$$

Centre of Mass of Solid of Revolution.

Let a solid be generated by the revolution of a curve or straight line about the axis of X. Any section made by a plane perpendicular to the axis of X will be circular. Let y be the radius of any section at a distance x from the origin. An element of volume will be $\pi y^2 dx$ where dx is the thickness of the disc.

The moment of this element about the origin will be

$$\pi x y^2 dx.$$

Hence

$$\bar{x}=\frac{\int \pi x y^2 dx}{\int \pi y^2 dx},$$

that is,

$$\bar{x}=\frac{\int x y^2 dx}{\int y^2 dx} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

EXAMPLE.

Find the centre of mass of a cone whose height is h and radius of its base is r.

Taking the vertex as origin and axis as the axis of X, the radius of a right section at a distance x from the origin is

$$\frac{rx}{h}=y,$$

therefore

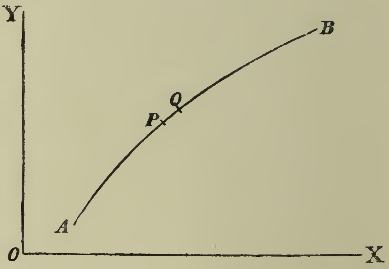
$$\bar{x} = \frac{\int_0^h x \frac{r^2 x^2}{h^2} d x}{\int_0^h \frac{r^2 x^2}{h^2} d x} = \frac{\int_0^h x^3 d x}{\int_0^h x^2 d x}.$$

Hence

$$\bar{x} = \frac{\left[\frac{x^4}{4}\right]_0^h}{\left[\frac{x^3}{3}\right]_0^h} = \frac{3}{4} h.$$

Centre of Mass of an Arc of any Uniform Plane Curve.

Let A B be an arc of a curve whose centre of mass we require. For simplicity we shall assume the mass of unit length to be unity.



Consider an element of arc P Q $= d s$. The moment of this element about the axis of Y is $x\, d s$. The whole arc A B is $\int d s$; therefore

$$\bar{x} \int d s = \int x\, d s,$$

that is,

$$\bar{x} = \frac{\int x\, d s}{\int d s} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Similarly,

$$\bar{y} = \frac{\int y\,d\,s}{\int d\,s} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

(1) and (2) may be written in the form

$$= \frac{\int x \sqrt{\left\{1 + \left(\frac{d\,y}{d\,x}\right)^2\right\}}\,d\,x}{\int \sqrt{\left\{1 + \left(\frac{d\,y}{d\,x}\right)^2\right\}}\,d\,x}, \quad \bar{y} = \frac{\int y \sqrt{\left\{1 + \left(\frac{d\,y}{d\,x}\right)^2\right\}}\,d\,x}{\int \sqrt{\left\{1 + \left(\frac{d\,y}{d\,x}\right)^2\right\}}\,d\,x}, \quad (3)$$

since $d\,s = \sqrt{(d\,x)^2 + (d\,y)^2} = \sqrt{\left\{1 + \left(\frac{d\,y}{d\,x}\right)^2\right\}}\;d\,x.$

Using polar co-ordinates we have, instead of x and y, $r\cos\theta$ and $r\sin\theta$ respectively.

Hence formulæ (3) become

$$\bar{x} = \frac{\int r\cos\theta \sqrt{\left\{r^2 + \left(\frac{d\,r}{d\,\theta}\right)^2\right\}}\;d\,\theta}{\int \sqrt{\left\{r^2 + \left(\frac{d\,r}{d\,\theta}\right)^2\right\}}\;d\,\theta} \quad . \quad . \quad . \quad (4)$$

$$\bar{y} = \frac{\int r\sin\theta \sqrt{\left\{r^2 + \left(\frac{d\,r}{d\,\theta}\right)^2\right\}}\;d\,\theta}{\int \sqrt{\left\{r^2 + \left(\frac{d\,r}{d\,\theta}\right)^2\right\}}\;d\,\theta} \quad . \quad . \quad . \quad (5)$$

EXAMPLES.

(1) Find the centre of area of the parabola $y^2 = 100\,x$ between the limits $x = 0$, $x = 16$.

Ans. $\bar{x} = \frac{3}{5} \times 16$, $\bar{y} = 15$.

(2) Find the centre of area of a quadrant of a circle.

Ans. $\bar{x} = \bar{y} = \frac{4\,r}{3\,\pi}$.

(3) Find the centre of area of the curve whose equation is $r = 12\,(1 + \cos\theta)$.

Ans. $\bar{x} = 10$.

(4) Find the centre of mass of a hemisphere of uniform density.

Ans. $\bar{x} = \frac{3\,r}{8}$.

(5) Find the centre of mass of a hemisphere whose density varies as the square of the distance from the centre.

Ans. $\bar{x} = \frac{5\,r}{12}$.

(6) Find the centre of area of a sector of a circle whose angle is $2\,\theta$.

Ans. $\bar{x} = \frac{2}{3}\,r\,\frac{\sin\theta}{\theta}$.

(7) Find the centre of mass of the paraboloid of revolution whose generating curve is given by the equation $y^2 = 4\,x$ between the limits $x = 0$ and $x = 30$.

Ans. $\bar{x} = 20$.

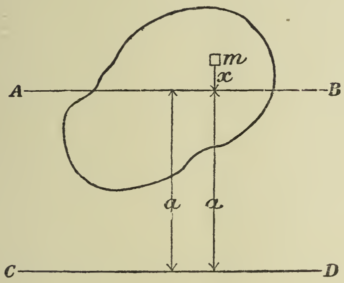
(8) Find the centre of mass of a frustum of a cone.

Theorem of Parallel Axes.

The moment of inertia of a body about any axis exceeds its moment of inertia about a parallel axis through its centre of mass by the product of the mass into the square of the distance between the parallel axes.

Proof.—In the accompanying figure let A B be an axis through the centre of mass, and let C D be a parallel axis at a distance a from A B. The moment of inertia of an element of mass m about C D is

$$m(x + a)^2.$$



The moment of inertia I of the whole mass about C D is

$$I = \Sigma m(x + a)^2 = \Sigma m x^2 + \Sigma 2 a m x + \Sigma a^2 m.$$

But $\Sigma m x^2$ is the moment of inertia I_0 of the mass about A B, and

$$\Sigma 2 a m x = 2 a \Sigma m x = 0;$$

also $\Sigma a^2 m = a^2 M$ where M is the whole mass; therefore

$$I = I_0 + a^2 M. \quad . \quad . \quad . \quad . \quad (1)$$

(1) is also true for an area if we substitute A the area instead of M,

$$I = I_0 + a^2 A. \quad . \quad . \quad . \quad . \quad (2)$$

EXAMPLES.

Find the moment of inertia of a circular area (1) about a tangent perpendicular to its plane; (2) about a tangent in its plane.

By Art. (210) the moment of inertia of a circular area about an axis through its centre perpendicular to its plan is $\frac{\pi r^4}{2}$, therefore by (2)

$$I = \frac{\pi r^4}{2} + \pi r^2 \times r^2 = \frac{3 \pi r^4}{2},$$

answer to (1).

$$I = \frac{\pi r^4}{4} + \pi r^2 \times r^2 = \frac{5 \pi r^4}{4},$$

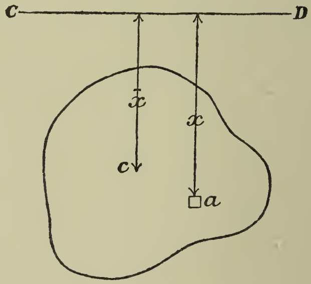
answer to (2).

Find the moment of inertia of a sphere about a tangent. By Art. (212) and (1)

$$I = \frac{2}{5} M r^2 + M r^2 = \frac{7}{5} M r^2.$$

Total Pressure on an Area immersed in a Liquid.

Let w be the weight of unit volume of the liquid, A the total area, a a very small portion of the area A, and let the area be inclined to the horizontal at an angle θ. Let the plane containing the area intersect the surface of the liquid in the line C D.



The pressure on an element of area a is proportional to the vertical depth below the surface, therefore the pressure on a, at a distance x from C D, is

$$w\,x\,a \sin\theta.$$

The total pressure is $\Sigma\, w\,x\,a \sin\theta$, that is

$$w \sin\theta\, \Sigma\, a\,x.$$

But by the principle of moments

$$w \sin\theta\, \Sigma\, a\,x = w \sin\theta\, A\,\bar{x},$$

where $\bar{x}$ is the distance of the centre of area from C D. This proves that the total pressure on an immersed area is equal to the product of the area, the depth of the centre of area below the surface and the weight of unit volume of the liquid.

EXAMPLE.

Find the total pressure on a circular area, 3 feet in diameter, the centre of the circle being 4 feet below the surface of water.

$$\text{Pressure} = \cdot7854 \times 3^2 \times 4 \times 62\cdot4 \text{ lbs.}$$

62·4 lbs. being the weight of a cubic foot of water.

Depth of Centre of Pressure on an immersed Area.

The pressure on an element of area a is

$$w\,a\,x \sin\theta,$$

and the moment of this pressure about the line C D is

$$w\,a\,x^2 \sin\theta.$$

The total moment is $\Sigma\, w\,a\,x^2 \sin\theta$, which is

$$w \sin\theta\, \Sigma\, a\,x^2.$$

But this is equal to the moment of the total pressure, about C D; that is

$$\overline{\mathrm{X}}\, w \sin\theta\, \mathrm{A}\, \bar{x} = w \sin\theta\, \Sigma\, a\, x^2;$$

therefore

$$\overline{\mathrm{X}} = \frac{w \sin\theta\, \Sigma\, a\, x^2}{w \sin\theta\, \mathrm{A}\, \bar{x}} = \frac{\Sigma\, a\, x^2}{\mathrm{A}\, \bar{x}},$$

where $\overline{\mathrm{X}}$ is the distance of the centre of pressure from the line C D; that is

$$\overline{\mathrm{X}} = \frac{\text{Moment of inertia of A about C D}}{\text{Distance of centre of area from C D} \times \text{A}}.$$

EXAMPLE.

Find the depth of the centre of pressure on a vertical circular area of radius 2 feet, whose centre is 6 feet below the surface.

$$\bar{x} = \frac{\frac{\pi r^4}{4} + \pi r^2 a^2}{\pi r^2 a} = \frac{\frac{r^2}{4} + a^2}{a},$$

where

$$r = 2 \text{ and } a = 6.$$

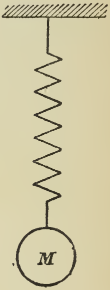
$$\therefore \quad \bar{x} = \frac{37}{6} = 6\tfrac{1}{6} \text{ feet} = 6 \text{ feet } 2 \text{ inches};$$

that is, the centre of pressure is 2 inches below the centre of the circle.

Vibrations, Natural and Forced.

A mass of M lb., suspended by a spiral spring whic elongates one foot when pulled by a force of μ poundals, wil continue to move up and down with a *simple harmonic motion* with undiminishing amplitude, provided there be no frictional resistance to the body's motion.

When a spiral spring is subjected to a pull or push along its axis, it is found that the displacement is proportional to the force exerted, a condition necessary to produce simple harmonic motion.



When the mass M is displaced from its mean position a distance of x feet, the restoring force is therefore μx poundals. Using the fundamental equation

Force (poundals) = acceleration × mass (lb.),

we have

$$\mathrm{M}\frac{d^2 x}{d t^2} = -\mu x, \quad . \quad . \quad . \quad . \quad (1)$$

since the expression for an acceleration is $\frac{d^2 x}{d t^2}$, where x is the displacement and t is the time in seconds reckoned from some instant. The negative sign on the right-hand side of (1) is accounted for by the fact that the restoring force acts in such a way as always to bring the body to its mean position.

From (1) we have, by dividing by M and transposing,

$$\frac{d^2 x}{d t^2} + \frac{\mu}{\mathrm{M}} x = 0. \quad . \quad . \quad . \quad . \quad (2)$$

The solution of (2) is got by assuming $x = k e^{mt}$, and forming the auxiliary equation, the roots of which are

$$m = \sqrt{\frac{\mu}{\mathrm{M}}}\sqrt{-1}, \quad \text{or,} \quad -\sqrt{\frac{\mu}{\mathrm{M}}}\sqrt{-1},$$

consequently the general solution to (2) is

$$x = k e^{\sqrt{\frac{\mu}{\mathrm{M}}}\sqrt{-1}\,t} + k_1 e^{\sqrt{\frac{\mu}{\mathrm{M}}}\sqrt{-1}\,t} \quad . \quad . \quad . \quad (3)$$

Now

$$e^{a\sqrt{-1}\,t} = \cos a t + \sqrt{-1} \sin a t,$$

and

$$e^{-a\sqrt{-1}\,t} = \cos a t - \sqrt{-1} \sin a t,$$

therefore the solution of (3) is

$$x = \text{A} \cos \sqrt{\frac{\mu}{\text{M}}}\, t + \text{B} \sin \sqrt{\frac{\mu}{\text{M}}}\, t. \quad . \quad . \quad (4)$$

If we reckon time from the instant the body is passing its mean position, we have

$$0 = \text{A} \cos 0^\circ + \text{B} \sin 0^\circ,$$

therefore

$$\text{A} = 0.$$

Hence

$$x = \text{B} \sin \sqrt{\frac{\mu}{\text{M}}}\, t. \quad . \quad . \quad . \quad . \quad (5)$$

Again, suppose the velocity 0 when $t = 0$, from (4) we have, by differentiating,

$$\frac{dx}{dt} = -\text{A} \sqrt{\frac{\mu}{\text{M}}} \sin \sqrt{\frac{\mu}{\text{M}}}\, t + \text{B} \sqrt{\frac{\mu}{\text{M}}} \cos \sqrt{\frac{\mu}{\text{M}}}\, t,$$

$$\therefore \quad 0 = \text{A} \times 0 + \text{B} \sqrt{\frac{\mu}{\text{M}}}. \qquad \therefore \quad \text{B} = 0.$$

Therefore (4) becomes

$$x = \text{A} \cos \sqrt{\frac{\mu}{\text{M}}\, t},$$

if we reckon time from the instant the body is at its maximum displacement A.

To find the time of one complete oscillation, we have

$$\frac{2\pi}{\text{T}} = \sqrt{\frac{\mu}{\text{M}}},$$

where T is the periodic time.

EXAMPLE.

Suppose M = 40 lbs. and let $\mu = 60$ poundals;

$$\therefore \quad \text{T} = 2 \times 3{\cdot}14 \times {\cdot}816 = 5{\cdot}12 \text{ seconds}.$$

Suppose the mass to vibrate in a resisting medium, whose frictional resistance is $2f$ poundals when the velocity is 1 foot per second, and suppose the friction proportional to the velocity, then the equation of motion is

$$\mathrm{M}\frac{d^2 x}{d t^2} + 2 f \frac{d x}{d t} + \mu x = 0 \quad . \quad . \quad . \quad (6)$$

The roots of the auxiliary equation of (6) are

$$m = -\frac{f}{\mathrm{M}} \pm \sqrt{\frac{f^2}{\mathrm{M}^2} - \frac{\mu}{\mathrm{M}}}\,.$$

First suppose

$$\frac{f^2}{\mathrm{M}^2} < \frac{\mu}{\mathrm{M}},$$

and let

$$a = \frac{f}{\mathrm{M}} \text{ and } \beta = \sqrt{\frac{\mu}{\mathrm{M}} - \frac{f^2}{\mathrm{M}^2}};$$

therefore

$$m = -a \pm \beta i, \text{ where } i = \sqrt{-1}.$$

Hence

$$x = k e^{(-a+\beta i)t} + k_1 e^{(-a-\beta i)t},$$

$$\therefore \quad x = e^{-at}\{\mathrm{A} \cos \beta t + \mathrm{B} \sin \beta t\} \quad . \quad . \quad . \quad (7)$$

where A and B are constants. (7) may be expressed in the form

$$x = \mathrm{C} e^{-at} \sin (\beta t + \theta) \quad . \quad . \quad . \quad (8)$$

$$\text{where } \mathrm{C} = \sqrt{\mathrm{A}^2 + \mathrm{B}^2}, \text{ and } \tan \theta = \frac{\mathrm{A}}{\mathrm{B}}.$$

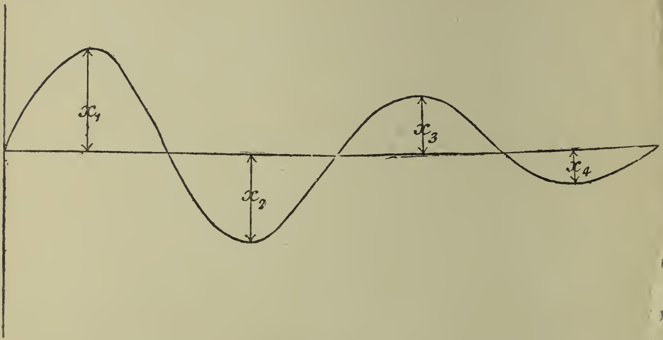
If x be plotted vertically, and t horizontally, we get a curve as shown in the accompanying figure.

Equation (8) takes different forms according to where we choose the vertical axis.

If we take the origin at the point where the curve cuts the time axis, equation (8) becomes

$$x = \mathrm{C} e^{-at} \sin \beta t \quad . \quad . \quad . \quad . \quad (9)$$

In a particular case the periodic time T was observed to be 1·2566 seconds, and the ratio of x_2 to x_1, x_3 to x_2, &c. was



·284; μ was 116 poundals and x_1 was ·1 foot. The mass M, frictional resistance $2f$ and C were found in the following way.

Equation (9) gives the displacement x_1 at any instant t_1 seconds. The displacement x_2 at time $t_1 + \delta t$ is

$$x_2 = C\, e^{-\alpha(t_1 + \delta t)} \sin \beta\, (t_1 + \delta t)\,;$$

that is,

$$x_2 = e^{-\alpha \delta t}\, C\, e^{-\alpha t_1} \sin \beta\, (t_1 + \delta t).$$

Similarly,

$$x_3 = e^{-2\alpha \delta t}\, C\, e^{-\alpha t_1} \sin \beta\, (t_1 + 2\,\delta t).$$

Let δt have such a value that

$$\beta\, \delta t = \pi,$$

therefore

$$\sin \beta\, t_1 = \sin \beta\, (t_1 + 2\,\delta t)\,;$$

therefore

$$\frac{x_3}{x_1} = e^{-2\alpha\delta t} = (\cdot 284)^2\,;$$

that is,

$$-\,2\,\alpha\,\delta t = 2\cdot 302 \times 2 \log. \cdot 284,$$

whence

$$\alpha\,\delta t = 1\cdot 2566.$$

Again, $\delta t = \frac{1}{2} \times$ periodic time $= \cdot 628$,

$$\text{and } \beta \, \delta t = \pi, \qquad \therefore \quad \beta = \frac{3 \cdot 1416}{\cdot 628} = 5.$$

Also $\alpha \, \delta t$, which is called the **Logarithmic decrement**, is $1 \cdot 2566$;

$$\therefore \quad \alpha = \frac{1 \cdot 2566}{\cdot 628} = 2.$$

Again,

$$\beta^2 = \frac{\mu}{M} - \frac{f^2}{M^2}, \qquad \text{and } \alpha = \frac{f}{M};$$

herefore

$$25 = \frac{116}{M} - 4, \text{ whence } M = 4 \text{ lbs.}$$

nd

$$f = 2\,M = 8 \text{ poundals.}$$

To get C, x_1 was found to be $\cdot 1$ foot, and the corresponding alue of t_1 measured from the origin was one-fourth of the eriodic time $= \frac{1 \cdot 2566}{4} = \cdot 314$,

$$\therefore \quad \cdot 1 = e^{-2 \times \cdot 314} \, C \sin \frac{\pi}{2},$$

whence $C = \cdot 1871$ foot, the maximum displacement, if there were no friction,

herefore

$$x = e^{-2t} \cdot 1871 \sin 5\,t.$$

Suppose f to gradually increase, consequently β will get smaller and smaller, but $\beta = \frac{2\,\pi}{T}$, where T is the periodic time, therefore T will get greater and greater as f increases, and when

$$\frac{f^2}{M^2} = \frac{\mu}{M};$$

that is, when the two roots of the auxiliary equation are equal, the solution is

$$x = (\mathrm{A} + \mathrm{B}\,t)\,e^{-\alpha t} \quad . \quad . \quad . \quad . \quad (10)$$

where A and B are constants depending upon given data. From (10) it is obvious that, as time goes on, x diminishes, but can never become negative, if A and B be positive; that is, the body approaches nearer and nearer to the position which it would occupy if there were no friction, but never reaches it.

The foregoing solutions apply equally as well to electric oscillations, see page 236.

Forced Vibrations.—In the foregoing investigations the upper end of the spiral spring was fixed. Now suppose the upper end of the spring to get a simple harmonic motion, whose amplitude is a and whose periodic time is T_1, and let us assume that there is no frictional resistance, the equation of motion is

$$\mathrm{M}\frac{d^2 x}{d\,t^2} + \mu\,(x - y) = 0 \quad . \quad . \quad . \quad (12)$$

where x is the displacement of the body from its mean position, and y is the displacement of the upper end of the spiral spring at any instant.

But $y = a \sin p\,t$, therefore (12) becomes

$$\mathrm{M}\frac{d^2 x}{d\,t^2} + \mu\,x = \mu\,a \sin p\,t \quad . \quad . \quad . \quad (13)$$

where $p = \dfrac{2\,\pi}{\mathrm{T}_1}$.

On dividing (13) by M and substituting n^2 for $\dfrac{\mu}{\mathrm{M}}$, and m for $\dfrac{\mu\,a}{\mathrm{M}}$, we have

$$\frac{d^2 x}{d\,t^2} + n^2 x = m \sin p\,t \quad . \quad . \quad . \quad (14)$$

A solution of this equation is

$$x = \mathrm{A} \sin p\,t + \mathrm{B} \cos p\,t \quad . \quad . \quad . \quad (15)$$

Differentiating (15) we have

$$\frac{d^2 x}{d t^2} = -p^2 \text{A} \sin p t - \text{B} p^2 \cos p t.$$

Substituting in (14) we get

$$-p^2 \text{A} \sin p t - \text{B} p^2 \cos p t + n^2 \text{A} \sin p t + n^2 \text{B} \cos p t = m \sin p t.$$

Equating the coefficients of sin $p t$, and also of cos $p t$ on both sides, we have

$$-p^2 \text{A} + n^2 \text{A} = m, \text{ and } -\text{B} p^2 + n^2 \text{B} = 0,$$

whence

$$\text{A} = \frac{m}{n^2 - p^2} \text{ and B} = 0;$$

therefore (15) becomes

$$x = \frac{m}{n^2 - p^2} \sin p t \quad . \quad . \quad . \quad (16)$$

Note that the amplitude of the motion of M increases as the value of p approaches that of n; and when p is equal to n the amplitude is infinity.

Relation between the Periodic Times of a Damped and an Undamped Vibration.

The general equation of the motion of a body vibrating without friction is

$$\frac{d^2 x}{d t^2} + n^2 x = 0.$$

A particular solution of which is

$$x = \text{A} \sin n t,$$

where

$$n = \frac{2 \pi}{\text{T}} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and T is the periodic time.

The equation of motion of the same body, where friction is considered, is

$$\frac{d^2 x}{d t^2} + 2 f \frac{d x}{d t} + n^2 x = 0;$$

a solution of which is

$$x = C e^{-ft} \sin \sqrt{n^2 - f^2}\, t,$$

consequently

$$\sqrt{n^2 - f^2} = \frac{2 \pi}{T_1} \quad . \quad . \quad . \quad . \quad (2)$$

where T_1 is the periodic time with friction.

From (1)

$$n^2 = \frac{4 \pi^2}{T^2},$$

and from (2)

$$n^2 = \frac{4 \pi^2}{T^2} + f^2;$$

therefore

$$\frac{4 \pi^2}{T^2} = \frac{4 \pi^2}{T_1^2} + f^2;$$

that is,

$$\frac{1}{T^2} = \frac{1}{T_1^2} + \frac{f^2}{4 \pi^2}.$$

This shows that the periodic time increases as the friction increases.

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